AAE 203 Notes

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Contents

Chapter 1

Introduction

HI!

In Mechanics, we are concerned with bodies which are at rest or in motion.

Kinematics. In kinematics we study motion without consideration of the causes of motion. This course is mainly concerned with kinematics of points.

Basic concepts: position, time

Derived concepts: velocity, acceleration; angle, angular velocity, angular acceleration

Statics Before full consideration of Dynamics, we look at Statics. Here bodies are ”at rest” and we examine the forces on the bodies.

Basic concept: force

Derived concepts: moment

Dynamics

Basic concept: mass

Basic Laws

Newtons First Law

Newtons Second Law

Newtons Third Law

Chapter 2

Units and Dimensions

2.1 Introduction

There are **four** fundamental quantities in mechanics:

$$\boxed{\text{length, time, mass, force}}$$

The first three are scalar quantities and the fourth is a vector quantity. All other quantities in mechanics can be derived from these fundamental quantities. For example, area is length by length, speed can be expressed as the ratio of length over time, and angle can be expressed as the ratio of length over length. Actually, the four fundamental quantities are not independent, they are related by Newton's second law. Hence one can choose any three of these quantities as basic quantities and consider the fourth as a derived quantity.

2.2 Units

When representing a physical quantity by a scalar or a vector, one must also specify units, for example,

$$l = 10\ \text{ft}\,.$$

The units of any quantity in mechanics can be expressed in terms of the units of any three of the four fundamental quantities. We will look at the two systems of units in common use, the SI system and the US system.

If a quantity is dimensionless its units are independent of the units chosen for the basic quantities. As we shall see shortly, one such quantity is angle. The two commonly used units for angles are radians and degrees. They are related by

$$180\ \text{degrees} = \pi\ \text{radians}$$

where π is the ratio of the circumference of any circle to its diameter; it is approximately given by

$$\pi \approx 3.146\,.$$

2.2.1 SI system of units

In the SI (or metric) system of units, the quantities mass, length and time are considered basic and force is derived.

quantity	unit	unit symbol
mass	kilogram	kg
length	meter	m
time	second	s
force	newton	N

As a consequence of Newton's second law, one newton is defined to be the magnitude of the force required to give 1 kg of mass an inertial acceleration of magnitude $1\mathrm{ms}^{-2}$, that is,

$$1\mathrm{N} = 1\mathrm{kg}\,\mathrm{m}\,\mathrm{s}^{-2}\,.$$

The units of any other quantity in mechanics can be expressed in terms of the units of the basic quantities, that is, kilograms, meters and seconds.

2.2.2 US system of units

In the US system of units, the quantities force, length and time are considered basic and mass is derived.

quantity	unit	unit symbol
force	pound	lb
length	foot	ft
time	second	sec
mass	slug	slug

As a consequence of Newton's second law, one slug is the mass which has an inertial acceleration of magnitude $1\mathrm{ft}\ \mathrm{sec}^{-2}$ when subject to a force of magnitude 1 lb, that is,

$$1\,\mathrm{lb} = 1\,\mathrm{slug}\,\mathrm{ft}\,\mathrm{sec}^{-2}\,.$$

Hence,

$$1\,\mathrm{slug} = 1\,\mathrm{lb}\,\mathrm{sec}^2\mathrm{ft}^{-1}$$

The units of any other quantity in mechanics can be expressed in terms of the units of the basic quantities, that is, pounds, feet and seconds.

2.2.3 Unit conversions

You should already know how to do this.

2.3 Dimensions

To every quantity in mechanics, we associate a dimension. Dimension indicates quantity type. We sometimes use symbols to indicate dimension. These symbols for the fundamental quantities are given in the following table.

quantity	dimension symbol
force	F
mass	M
length	L
time	T

Note that the concept of dimension is not the same as unit. One foot is not the same as one meter, however, both have the same dimension, namely, length.

2.3.1 Dimensional systems

The dimensions of the four fundamental quantities are related by Newton's second law, specifically,

$$F = MLT^{-2}.$$

Hence we can choose any three dimensions as basic dimensions and consider the fourth dimension as a derived dimension. Usually, one chooses M, L, T or F, L, T as basic dimensions.

Absolute dimensional system. In the absolute dimensional system, mass, length and time are considered basic and force is derived. The dimension of any quantity in mechanics is expressed as

$$\boxed{M^\alpha L^\beta T^\gamma}$$

where α, β and γ are real numbers. For example, $F = MLT^{-2}$.

Gravitational dimensional system. In the gravitational dimensional system, force, length and time are considered basic and mass is derived. The dimension of any quantity in mechanics is expressed as

$$\boxed{F^\alpha L^\beta T^\gamma}$$

where α, β and γ are real numbers. For example, $M = FL^{-1}T^2$.

2.4 Dimensions of derived quantities

The dimension of any quantity Q in mechanics can be obtained using the following simple rules. We will use the notation $\dim[Q]$ to indicate the dimension of quantity Q. The dimension of a vector quantity $\bar{Q}$ is considered to be the same as that of its magnitude, that is, $\dim[\bar{Q}] = \dim[|\bar{Q}|]$.

Dimensions of numbers. A "pure" number Q is considered dimensionless. We indicate this by

$$\dim[Q] = 1$$

Dimensions of products and quotients. If Q_1 and Q_2 are any two quantities, then

$$\dim[Q_1 Q_2] = \dim[Q_1] \dim[Q_2] \qquad \text{and} \qquad \dim[Q_1/Q_2] = \dim[Q_1]/\dim[Q_2] \,.$$

Example 1 (Angle) In radians, the angle θ is given by $\theta = S/R$. Since S and R are

Figure 2.1: Angle

lengths, we have

$$\dim[\theta] = \dim[S/R] = \dim[S]/\dim[R] = L/L = 1 \,.$$

Since $\dim[\theta] = 1$, we consider angles dimensionless.

Example 2 (cos and sin) Since $\cos\theta = a/c$, where a and c and lengths, we have

$$\dim[\cos\theta] = \dim[a/c] = \dim[a]/\dim[c] = L/L = 1 \,.$$

In a similar fashion,

$$\dim[\sin\theta] = \dim[b/c] = \dim[b]/\dim[c] = L/L = 1 \,.$$

and

$$\dim[\tan\theta] = \dim[b/a] = \dim[b]/\dim[a] = L/L = 1 \,.$$

Dimensions of powers. If Q is any quantity and α is any real number, then

$$\dim[Q^\alpha] = \dim[Q]^\alpha \,.$$

Figure 2.2: cos, sin and tan

Example 3 What is the dimension of the quantity $Q = \sqrt{gh}$ where h represents a height and g is a gravitational acceleration constant?

Since $\sqrt{gh} = [gh]^{\frac{1}{2}}$, we can use the power and product rules to first obtain that

$$\dim\left[\sqrt{gh}\right] = \dim\left[(gh)^{\frac{1}{2}}\right] = (\dim[g]\dim[h])^{\frac{1}{2}} .$$

Since h represents a height, we have $\dim[h] = L$; since g is an acceleration we also have $\dim[g] = LT^{-2}$. Thus

$$\dim\left[\sqrt{gh}\right] = [(LT^{-2})(L)]^{\frac{1}{2}} = LT^{-1} .$$

Notice that $\sqrt{gh}$ has the dimension of speed.

Dimensions of sums. It does not make sense to add quantities of different dimensions, so, we have the following rule:

Only quantities of the same dimensions should be added or subtracted.

Thus, if Q_1 and Q_2 are are two quantities of the same dimension, then

$$\dim[Q_1 + Q_2] = \dim[Q_1] = \dim[Q_2] \qquad \text{and} \qquad \dim[Q_1 - Q_2] = \dim[Q_1] = \dim[Q_2] .$$

Dimensions and derivatives.

$$\dim\left[\frac{dy}{dx}\right] = \dim[y]/\dim[x]$$

Since

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) ,$$

it follows from two applications of the above rule that

$$\dim\left[\frac{d^2y}{dx^2}\right] = \dim\left[\frac{dy}{dx}\right] / \dim[x] = (\dim[y]/\dim[x]) / \dim[x] ,$$

that is,

$$\dim\left[\frac{d^2y}{dx^2}\right] = \dim[y]/\dim[x]^2 .$$

Dimensions and integrals.

$$\dim\left[\int y\,dx\right] = \dim[y]\,\dim[x]$$

Dimensions and equations. We say that an equation is dimensionally homogeneous if every term in the equation has the same dimension. We have the following rule:

All equations (in mechanics) must be dimensionally homogeneous.

Example 4 The expression for planar acceleration in polar coordinates is given by

$$\bar{a} = (\ddot{r} + r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

where $\hat{e}_r$ and $\hat{e}_\theta$ are dimensionless unit vectors. Let us check to see if every term in this equation has the dimension of acceleration, that is, LT^{-2}.

Example 5 Later we shall meet the inverse square gravitational law which is expressed as

$$F = \frac{GMm}{r^2}$$

where F is a force magnitude, M and m are masses while r is a distance. Here we shall determine the dimension of G.

2.5 Exercises

Exercise 1 Obtain expressions for the dimensions of the following quantities using (a) the absolute dimensional system, and (b) the gravitational dimensional system. Here x and y are lengths, t is time, m is some mass, a is an acceleration and F represents a force.

(a) $-\sqrt{10\int_{x_1}^{x_2} a\,dx}$

(b) $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

(c) $\frac{d^2}{dt^2}\left(\int_0^{x(t)} F(\eta)\,d\eta\right)$

Exercise 2 Determine whether or not the equation

$$\frac{d}{dt}\int_0^x F\,dx = \frac{1}{2}\frac{dm}{dt}v^2 + mva$$

is dimensionally homogeneous where F is a force, x is a displacement, v is a speed, a is an acceleration, m is some mass, and t is time.

Exercise 3 If m denotes a mass, g an acceleration magnitude, x a length, F a force magnitude and t time, determine whether or not the following equation is dimensional homogeneous.

$$mgx = \int_0^x F\,d\eta + m\left(\frac{dx}{dt}\right)^2 + \frac{d^2x}{dt^2}$$

If not homogeneous, state why.

Exercise 4 You have just spent the whole evening deriving the following expression for an acceleration in an AAE 203 problem:

$$\bar{a} = (l\ddot{\theta} + d\theta\Omega)\hat{s}_1 + d\dot{\Omega}\dot{\theta}\hat{s}_2$$

where l and d represent lengths, θ represents an angle, and Ω represents a rotation rate. Your roommate looks at the expression and without doing any kinematic calculations, says you are wrong. Could she/he be right? Justify your answer.

Exercise 5 Determine the dimension of h in order for the following equation to be dimensionally correct.

$$\ddot{\theta} + \frac{h}{l}\sin\theta = 0$$

where θ represents an angle and l represents a length.

Exercise 6 Given that F is a force, x is a displacement, θ is an angle, and v is a speed, determine the dimensions of the quantities I and k in order that the following equation be dimensionally homogeneous.

$$\int_0^x F\,dx = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}kv^2$$

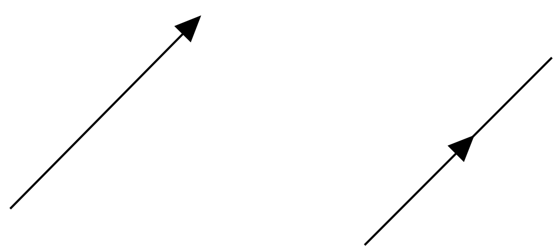
Chapter 3

Vectors

3.1 Introduction

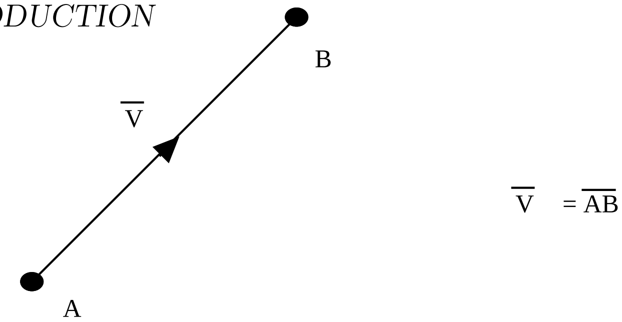
A scalar is a real number, for example, 1, $-1/2$, $\sqrt{2}$. Some physical quantities can be represented by a single scalar, for example, time, length, and mass. These quantities are called scalar quantities.

Other physical quantities cannot be represented by a single scalar, for example, force and velocity. These quantities have attributes of magnitude and direction. In saying that a motorcycle is traveling south at a speed of 70 mph, we are specifying the velocity of the cycle in terms of magnitude (70 mph) and direction (south). To represent velocity, we need a mathematical concept which has the above two attributes. A vector is such a concept. Mathematically, we define a vector to be a directed line segment. Graphically, we usually indicate a vector by a line segment with an arrowhead, for example,

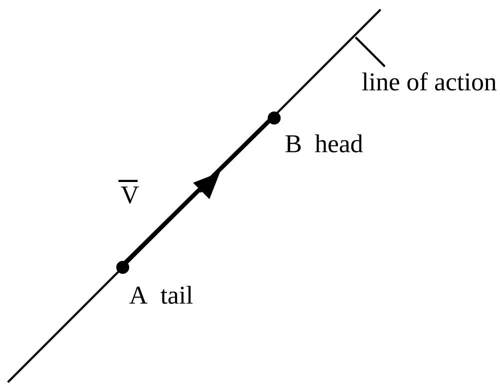


The magnitude of a vector is the length of the line segment while the direction of the vector is determined by the orientation of the line segment and the sense of the arrowhead. Sometimes a vector is indicated by a segment of a circle with an arrowhead; in this case the direction of the vector is determined by the right-hand rule. In the figure below, the direction of each vector is perpendicular to and out of the page.

All vectors, except unit vectors (we will meet these later), are represented by a symbol with an overhead bar, for example, $\bar{V}$, $\bar{0}$, $\bar{\clubsuit}$. If A and B are the endpoints of a vector $\bar{V}$ and the direction of $\bar{V}$ is from A to B, we sometimes write $\bar{V} = \overline{AB}$.

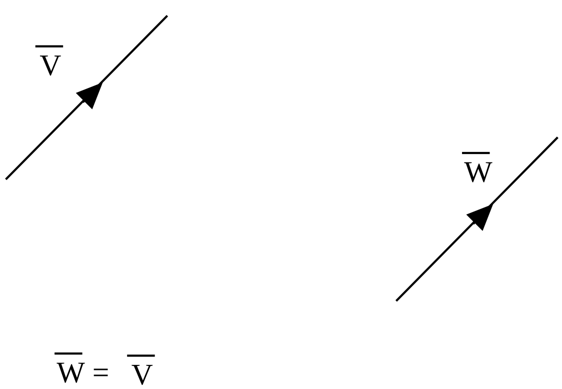


If $\bar{V} = \bar{A}B$, we call A the tail or point of application of $\bar{V}$ and B is called the head of $\bar{V}$. The line on which $\bar{V}$ lies is called the line of action of $\bar{V}$.



The magnitude or length of a vector $\bar{V}$ (denoted $|\bar{V}|$ and sometimes by V) is the distance between the endpoints of $\bar{V}$. Two vectors $\bar{V}$ and $\bar{W}$ are said to be parallel (denoted $\bar{V}$ // $\bar{W}$), if the lines of action of $\bar{V}$ and $\bar{W}$ are parallel.

Two vectors $\bar{V}$, $\bar{W}$, are defined to be equal, that is, $\bar{W} = \bar{V}$, if they have the same magnitude and direction. Thus one can completely specify a vector by specifying its magnitude and direction.



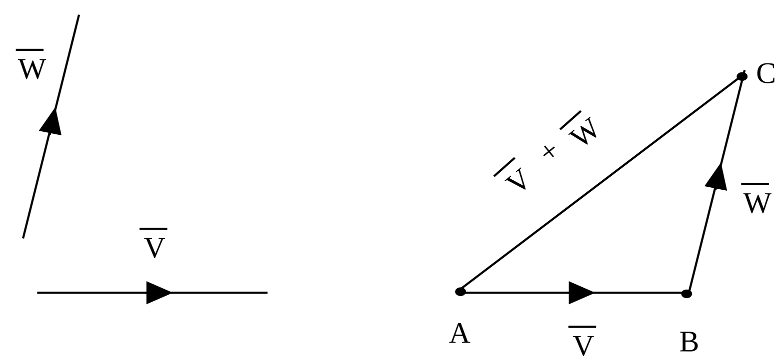
3.2 Vector addition

The sum or resultant of two vectors $\bar{V}$ and $\bar{W}$ is denoted by

$$\boxed{\bar{V} + \bar{W}}$$

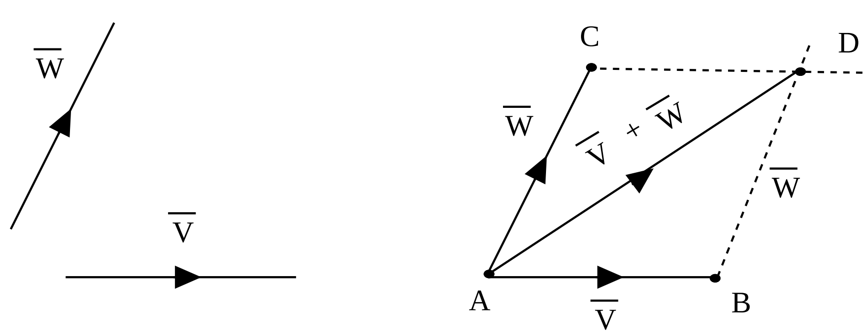
We present two equivalent definitions of vector addition, the triangle law and the parallelogram rule.

Triangle law. Place the tail of $\bar{W}$ at the head of $\bar{V}$. Then $\bar{V} + \bar{W}$ is the vector from tail of $\bar{V}$ to the head of $\bar{W}$.



In other words, if $\bar{V} = \overline{AB}$ and $\bar{W} = \overline{BC}$, then $\bar{V} + \bar{W} = \overline{AC}$.

Parallelogram rule. Place the tails of $\bar{V}$ and $\bar{W}$ together. Complete the parallelogram with sides parallel to $\bar{V}$ and $\bar{W}$. Then $\bar{V} + \bar{W}$ lies along a diagonal of the parallelogram with tail at the tails of $\bar{V}$ and $\bar{W}$.



In other words, if $\bar{V} = \overline{AB}$ and $\bar{W} = \overline{AC}$, then $\bar{V} + \bar{W} = \overline{AD}$ where $ABDC$ is a parallelogram.

One may readily show that the above two definitions are equivalent.

Some trigonometry Recall

angle
sine
cosine

Pythagorean theorem

$$c^2 = a^2 + b^2$$

Cosine law:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Sine Law

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Example 6 (Vector addition, cosine law, sine law.)

Given two vectors $\bar{V}$ and $\bar{W}$ as shown with $|\bar{V}| = 1$ and $|\bar{W}| = 2$. *Find* $\bar{V} + \bar{W}$.

Solution. We use the triangle law for vector addition as illustrated.

Using the cosine law on the above triangle, we obtain that

$$\begin{aligned} |\bar{V} + \bar{W}|^2 &= |\bar{V}|^2 + |\bar{W}|^2 - 2|\bar{V}|\ |\bar{W}| \cos(120°) \\ &= (1)^2 + (2)^2 - 2(1)(2)\left(-\frac{1}{2}\right) = 7\,. \end{aligned}$$

Hence,

$$|\bar{V} + \bar{W}| = \sqrt{7}\,.$$

Applying the sine law to the above triangle yields

$$\frac{\sin\theta}{|\bar{W}|} = \frac{\sin(120°)}{|\bar{V} + \bar{W}|}\,.$$

Hence,

$$\sin\theta = \frac{\sin(120°)|\bar{W}|}{|\bar{V} + \bar{W}|} = \frac{(\sqrt{3}/2)(2)}{\sqrt{7}} = \sqrt{\frac{3}{7}}$$

which yields

$$\theta = \sin^{-1}\left(\sqrt{\frac{3}{7}}\right) = 40.89°\,.$$

So,

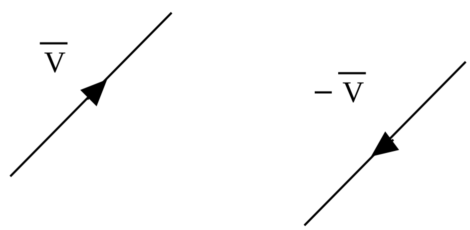
$$\boxed{\bar{V} + \bar{W} \text{ is a vector of magnitude } \sqrt{7} \text{ with direction as shown where } \theta = 40.89^\circ}$$

(Recall that a vector can be completely specified by specifying its magnitude and direction.)

• A zero vector is a vector of zero magnitude.

$$\cdot \; \bar{0}$$

• The negative of $\bar{V}$ (denoted $-\bar{V}$) is a vector which has the same magnitude as $\bar{V}$ but opposite direction.



3.2.1 Basic properties of vector addition

1) (Commutativity.) For every pair $\bar{V}$, $\bar{W}$ of vectors, we have

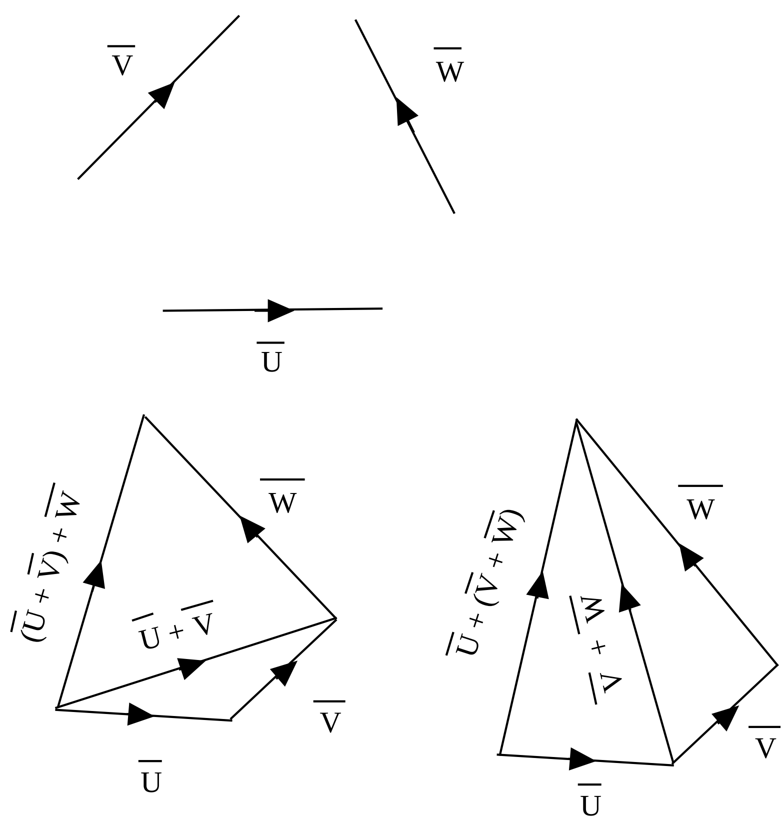
$$\bar{V} + \bar{W} = \bar{W} + \bar{V} \, .$$

(This follows from parallelogram rule)

2) (Associativity.) For every triplet $\bar{U}$, $\bar{V}$, $\bar{W}$ of vectors, we have

$$(\bar{U} + \bar{V}) + \bar{W} = \bar{U} + (\bar{V} + \bar{W})$$

.



3) There is a vector $\bar{0}$ such that for every vector $\bar{V}$ we have

$$\bar{V} + \bar{0} = \bar{V} \, .$$

4) For every vector $\bar{V}$ there is a vector $-\bar{V}$ such that

$$\bar{V} + (-\bar{V}) = \bar{0}$$

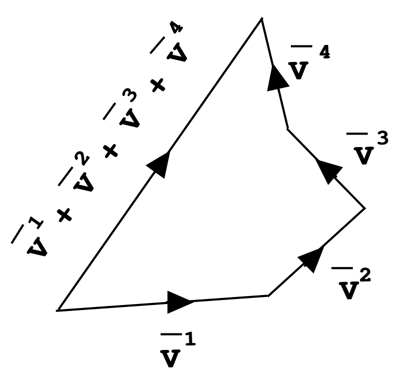
The above four properties are called the **group properties** of vector addition. The next property follows from the triangle law. It is called the **triangle inequality.**

5) For any pair $\bar{V}$, $\bar{W}$ of vectors, we have

$$|\bar{V} + \bar{W}| \;\leq\; |\bar{V}| + |\bar{W}|\,.$$

3.2.2 Addition of several vectors

Several vectors are added in the following fashion. Starting with the second vector, one simply places the tail of the vector at the head of the preceding vector. The sum of all the vectors is the vector from the tail of the first vector to the head of the last vector.



$$\bar{V} = \bar{V}^1 + \bar{V}^2 + \bar{V}^3 + \bar{V}^4$$

3.2.3 Subtraction of vectors

The difference of two vectors $\bar{V}$ and $\bar{W}$ is denoted by $\bar{V} - \bar{W}$ and is defined by

$$\bar{V} - \bar{W} = \bar{V} + (-\bar{W})\,.$$

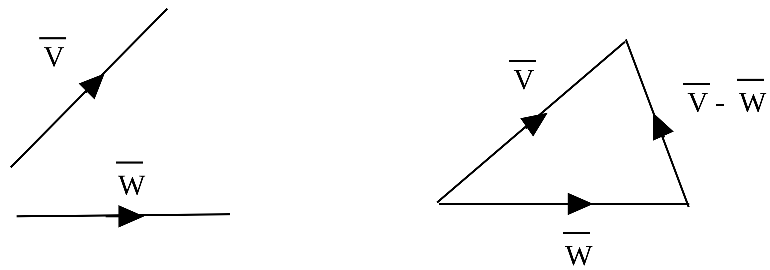


Figure 3.1: Subtraction of vectors

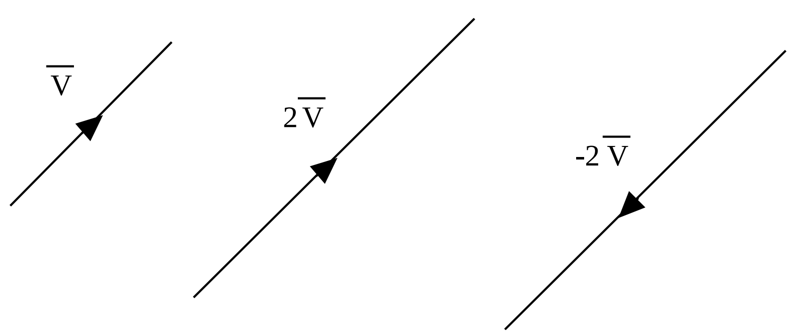
Note that, if the tails of $\bar{V}$ are $\bar{W}$ are placed together then, $\bar{V} - \bar{W}$ is the vector from the head of $\bar{V}$ to the head of $\bar{W}$.

3.3 Multiplication of a vector by a scalar

The product of a vector $\bar{V}$ and a scalar k is another vector and is denoted by

$$\boxed{k\bar{V}}$$

The product $k\bar{V}$ is defined to be a vector whose magnitude is $|k||\bar{V}|$. If $k > 0$, the direction of $k\bar{V}$ is the same as $\bar{V}$; if $k < 0$ the direction of $k\bar{V}$ is opposite to that of $\bar{V}$. If $k = 0$, the product $k\bar{V}$ is zero.



Basic properties of scalar multiplication

1) $k(\bar{V} + \bar{W}) = k\bar{V} + k\bar{W}$

2) $(k + l)\bar{V} = k\bar{V} + l\bar{V}$

3) $1\bar{V} = \bar{V}$

4) $k(l\bar{V}) = (kl)\bar{V}$

The above four properties along with the group properties of vector addition are called the field properties of vectors.

The next property is actually part of the above definition of scalar multiplication.

5) $|k\bar{V}| = |k||\bar{V}|$

3.3.1 Unit vectors

A unit vector is a vector of magnitude one. In writing unit vectors, we use "hats" instead of bars, for example, $\hat{u}$ represents a unit vector; hence $|\hat{u}| = 1$. Unit vectors are useful for indicating direction. If $\bar{V}$ is nonzero, the vector

$$\boxed{\hat{u}_{\bar{V}} \;:= \frac{\bar{V}}{|\bar{V}|}}$$

is called the unit vector in the direction of $\bar{V}$. Clearly, $\hat{u}_{\bar{V}}$ has the same direction as $\bar{V}$ and one can readily see that $\hat{u}_{\bar{V}}$ is a unit vector as follows.

$$|\hat{u}_{\bar{V}}| = \left|\frac{\bar{V}}{|\bar{V}|}\right| \; = \left|\frac{1}{|\bar{V}|}\right| |\bar{V}| = \frac{|\bar{V}|}{|\bar{V}|} = 1.$$

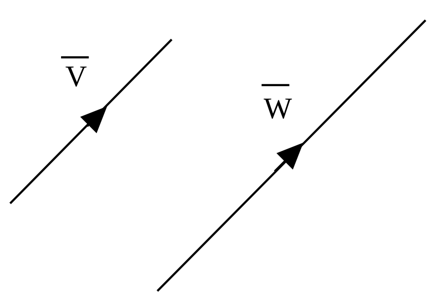
Note that

$$\boxed{\bar{V} = |\bar{V}| \; \hat{u}_{\bar{V}}}$$

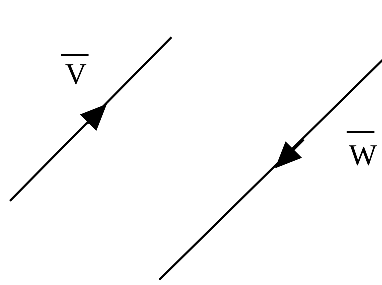
This explicitly represents a vector in terms of its magnitude $|\bar{V}|$ and its direction, the direction being completely specified by $\hat{u}_{\bar{V}}$.

Some useful facts. The following facts are useful for representing physical quantities by vectors. Suppose $\bar{V}$ and $\bar{W}$ are any two nonzero vectors. Then the following hold.

1) If $\bar{V}$ and $\bar{W}$ have the same direction, then there is a scalar $k > 0$ such that $\bar{W} = k\bar{V}$.



2) If the direction of $\bar{W}$ is opposite to the direction of $\bar{V}$, then there is a scalar $k < 0$ such that $\bar{W} = k\bar{V}$.



3) If $\bar{W}$ is parallel to $\bar{V}$, then there is a nonzero scalar k such that $\bar{W} = k\bar{V}$.

We now demonstrate why the above facts are true.

1) Since $\bar{W}$ and $\bar{V}$ have the same direction, the unit vector in the direction of $\bar{W}$ is equal to the unit vector in the direction of $\bar{V}$, that is,

$$\frac{\bar{W}}{|\bar{W}|} = \frac{\bar{V}}{|\bar{V}|} \; ;$$

hence,

$$\boxed{\bar{W} = \frac{|\bar{W}|}{|\bar{V}|}\,\bar{V}}$$

or,

$$\bar{W} = k\bar{V} \qquad \text{where} \qquad k = \frac{|\bar{W}|}{|\bar{V}|} \quad > \quad 0\,.$$

2) In this case, $-\bar{W}$ has the same direction as $\bar{V}$. Using the previous result, there is a scalar $l > 0$ such that

$$-\bar{W} = l\bar{V}\,.$$

Letting $k := -l$, we have

$$\bar{W} = k\bar{V} \qquad \text{with} \qquad k < 0\,.$$

3) Since $\bar{W}$ is parallel to $\bar{V}$, either $\bar{W}$ and $\bar{V}$ have the same direction or they have opposite direction. Hence, using the previous two results,

$$\bar{W} = k\bar{V} \qquad \text{where} \qquad k \;<\; 0 \quad \text{or} \quad k \;>\; 0.$$

3.4 Components

So far, our concept of a vector is a geometrical one, specifically, it is a mathematical object with the properties of magnitude and direction. This representation is useful for initial representation of physical quantities, for example, suppose one wants to describe the velocity of a motorcycle heading south at 70 mph as a vector. However, in manipulating vectors (for example adding them) the geometric representation can become very cumbersome if not impossible. In this section, we learn how to represent any vector as an ordered triplet of scalars, for example $(1, 2, 3)$. This permits us to reduce operations on vectors to operations on scalars.

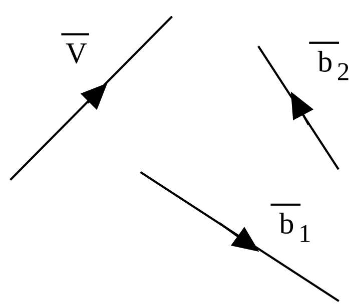
3.4.1 Planar case

Consider first the case in which all vectors of interest lie in a single plane,

Fact 1 *Suppose $\bar{b}_1, \bar{b}_2$, are any pair of non-zero, non-parallel vectors in a plane. Then, for every vector $\bar{V}$ in the plane, there is a unique pair of scalars, V_1, V_2 such that*

$$\boxed{\bar{V} = V_1\,\bar{b}_1 + V_2\,\bar{b}_2}$$

The pair $(\bar{b}_1, \bar{b}_2)$ of vectors is called a *basis*. It defines a *coordinate system*. With respect to this basis, $V_1\bar{b}_1$ and $V_2\bar{b}_2$ are called the *vector components* of $\bar{V}$; the scalars V_1 and V_2 are called the *scalar components* or *coordinates* of $\bar{V}$. The important thing about a basis is that it permits one to represent uniquely any vector $\bar{V}$ in the plane as a pair of scalars $(V_1,\ V_2)$.



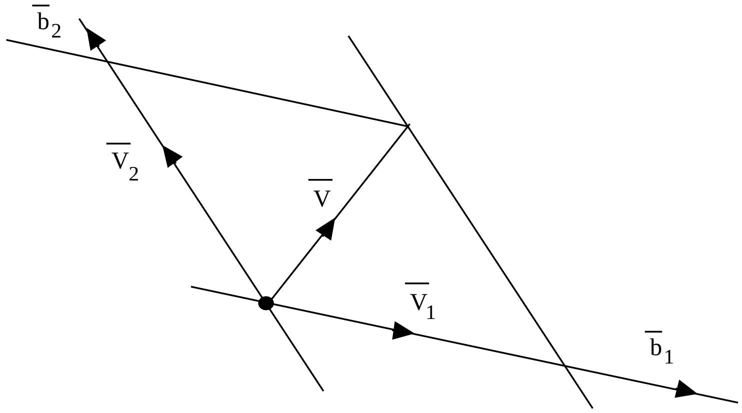
Demonstration of Fact 1. Construct a parallelogram with $\bar{V}$ as diagonal and with sides parallel to $\bar{b}_1$ and $\bar{b}_2$. Let $\bar{V}_1$ and $\bar{V}_2$ be the vectors with tails at the tail of $\bar{V}$ which make up two sides of the parallelogram as shown. Then $\bar{V}_1$ is parallel to $\bar{b}_1$, $\bar{V}_2$ is parallel to $\bar{b}_2$ and from the parallelogram law

$$\bar{V} = \bar{V}_1 + \bar{V}_2$$

Also, since $\bar{V}_1$ is parallel to $\bar{b}_1$ and $\bar{V}_2$ is parallel to $\bar{b}_2$, there are unique scalars V_1, V_2 so that

$$\bar{V}_1 = V_1\bar{b}_1 \qquad \text{and} \qquad \bar{V}_2 = V_2\bar{b}_2 \ .$$

Thus, $\bar{V}$ may be written as $\bar{V} = V_1\bar{b}_1 + V_2\bar{b}_2$. ∎



Example 7 (Planar components.) Given the coplanar vectors $\bar{V}$, $\hat{b}_1$, $\hat{b}_2$ as shown where $|\bar{V}| = 5$, *find*

(i) scalars V_1 and V_2 such that $\bar{V} = V_1\hat{b}_1 + V_2\hat{b}_2$,

(ii) scalars n_1 and n_2 such that $\hat{u}_{\bar{V}} = n_1\hat{b}_1 + n_2\hat{b}_2$.

Solution.
(i)

From the above parallelogram, it should be clear that

$$\bar{V} = \bar{V}_1 + \bar{V}_2\,.$$

Also, $\theta = 180 - 60 - 45 = 75°$. Using the sine law, we obtain that

$$\frac{\sin\theta}{|\bar{V}_1|} = \frac{\sin 45°}{|\bar{V}|}\,.$$

Hence,

$$|\bar{V}_1| = \frac{\sin(75°)|\bar{V}|}{\sin 45°} = \frac{(0.9659)(5)}{\frac{1}{\sqrt{2}}} = 6.830\,.$$

So, $\bar{V}_1 = |\bar{V}_1|\,\hat{b}_1 = 6.830\,\hat{b}_1$. Using the sine law again,

$$\frac{\sin 60°}{|\bar{V}_2|} = \frac{\sin 45°}{|\bar{V}|}\,.$$

Hence,

$$|\bar{V}_2| = \frac{\sin 60°}{\sin 45°}|\bar{V}| = \frac{\left(\frac{\sqrt{3}}{2}\right)(5)}{\frac{1}{\sqrt{2}}} = \frac{5\sqrt{3}}{\sqrt{2}} = 6.124\,.$$

So, $\bar{V}_2 = |\bar{V}_2|\,\hat{b}_2 = 6.124\,\hat{b}_2$ and

$$\boxed{\bar{V} = 6.830\,\hat{b}_1 + 6.124\,\hat{b}_2}$$

Note that, in this example, $|\bar{V}_1|,\ |\bar{V}_2| > |\bar{V}|$.

(ii) Since,

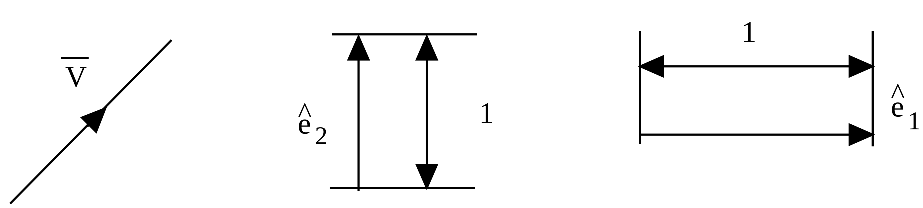
$$\hat{u}_{\bar{V}} = \frac{\bar{V}}{|\bar{V}|} = \frac{1}{5}(6.830\,\hat{b}_1 + 6.124\,\hat{b}_2)$$

we obtain that

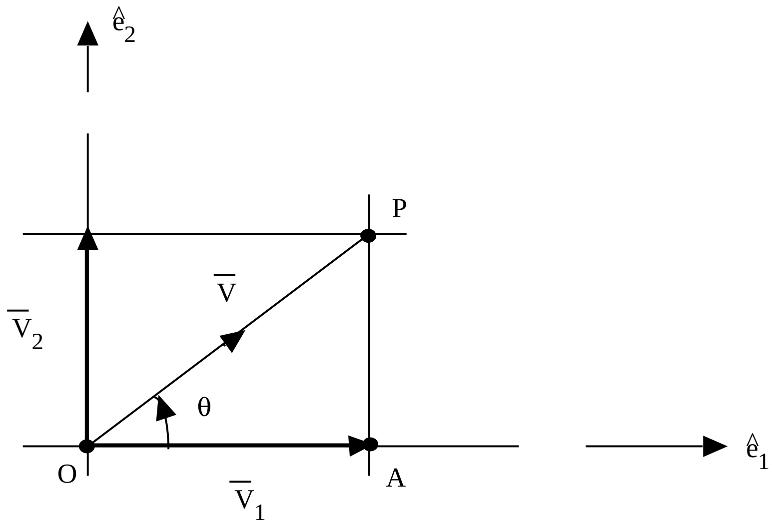
$$\boxed{\hat{u}_{\bar{V}} = 1.366\,\hat{b}_1 + 1.225\,\hat{b}_2}$$

Perpendicular components

Consider the special case in which $\bar{b}_1 = \hat{e}_1,\ \bar{b}_2 = \hat{e}_2$ and $\hat{e}_1,\ \hat{e}_2$ are mutually perpendicular unit vectors.



Then, the parallelogram used in obtaining components $\bar{V}_1$ and $\bar{V}_2$ is a rectangle and the components are sometimes called rectangular components.



From the parallelogram law of vector addition, it follows from the above figure that

$$\bar{V} = \bar{V}_1 + \bar{V}_2$$

Since the vector $\bar{V}_1$ is parallel to $\hat{e}_1$, we must have $\bar{V}_1 = V_1\hat{e}_1$ for some scalar V_1. In a similar fashion, $\bar{V}_2 = V_2\hat{e}_2$ where V_2 is a scalar. Hence,

$$\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 \,.$$

Using the Pythagorean theorem on triangle OAP we obtain that $|\bar{V}|^2 = |\bar{V}_1|^2 + |\bar{V}_2|^2$; hence

$$V = \sqrt{|\bar{V}_1|^2 + |\bar{V}_2|^2}$$

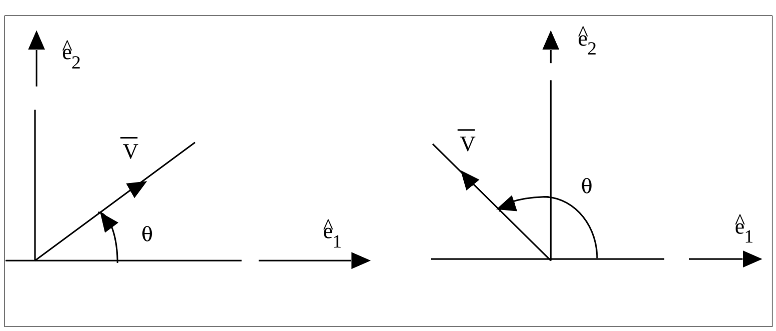
where $V := |\bar{V}|$ is the magnitude of $\bar{V}$. Since $|\bar{V}_1| = |V_1|$ and $|\bar{V}_2| = |V_2|$, we have

$$V = \sqrt{V_1^2 + V_2^2} \qquad .$$

For the situation illustrated,

$$V_1 = |\bar{V}_1| = V\cos\theta, \qquad V_2 = |\bar{V}_2| = V\sin\theta.$$

However, the relationships $V_1 = V\cos\theta$ and $V_2 = V\sin\theta$ hold for any direction of $\bar{V}$.



Summarizing, we have the following relationships:

$$\boxed{\begin{aligned} \bar{V} &= V_1\hat{e}_1 + V_2\hat{e}_2 \\ V_1 &= V\cos\theta, \qquad V_2 = V\sin\theta \end{aligned}}$$

Also,

$$\boxed{\begin{aligned} V &= \sqrt{V_1^2 + V_2^2} \\ \tan\theta &= V_2/V_1 \end{aligned}}$$

3.4.2 General case

Suppose $\bar{b}_1, \bar{b}_2$ and $\bar{b}_3$ are any three non-zero vectors which are not parallel to a common plane. Then, given any vector $\bar{V}$, there exists a unique triplet of scalars, V_1, V_2, V_3 such that

$$\bar{V} = V_1\bar{b}_1 + V_2\bar{b}_2 + V_3\bar{b}_3 \,.$$

The triplet of vectors, $(\bar{b}_1,\ \bar{b}_2,\ \bar{b}_3)$, is called a basis. It defines a coordinate system. With respect to this basis, the vectors $V_1\bar{b}_1, V_2\bar{b}_2, V_3\bar{b}_3$ are the vector components of $\bar{V}$ and the scalars V_1, V_2, V_3 are the scalar components or coordinates of $\bar{V}$. The most important thing about a basis is that it permits one to represent uniquely any vector $\bar{V}$ as a triplet of scalars (V_1, V_2, V_3). In this course, we consider mainly a special case, namely the case in which $\bar{b}_1, \bar{b}_2, \bar{b}_3$ are mutually perpendicular unit vectors.

Mutually perpendicular components

Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be any three mutually orthogonal (perpendicular) unit vectors. We call $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ an orthogonal triad. Since $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ constitute a basis, any vector $\bar{V}$ can be uniquely resolved

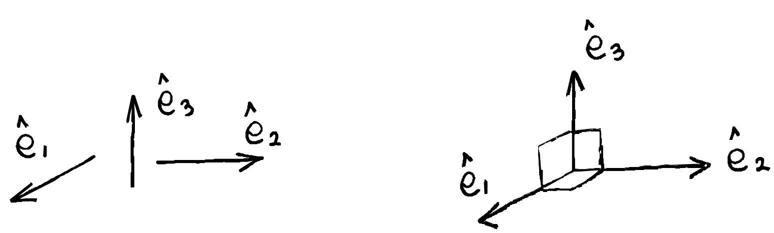


Figure 3.2: An orthogonal triad

into components parallel to $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, that is, there are unique scalars V_1, V_2, V_3 such that

$$\boxed{\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3}$$

The vectors $V_1\hat{e}_1, V_2\hat{e}_2, V_3\hat{e}_3$ are called rectangular components and the scalars V_1, V_2, V_3 are called rectangular scalar components or rectangular coordinates. Also,

$$\boxed{V = \sqrt{V_1^2 + V_2^2 + V_3^2}}$$

where $V = |\bar{V}|$.

To demonstrate the above decomposition, we first decompose $\bar{V}$ into two components, $\bar{V}_3$ and $\bar{V}_I$ where $\bar{V}_3$ is parallel to $\hat{e}_3$ and $\bar{V}_I$ is in the plane formed by $\hat{e}_1$ and $\hat{e}_2$. Thus,

$$\bar{V} = \bar{V}_I + \bar{V}_3 \,.$$

Also, using the Pythagorean theorem, we have

$$|\bar{V}|^2 = |\bar{V}_I|^2 + |\bar{V}_3|^2 \,.$$

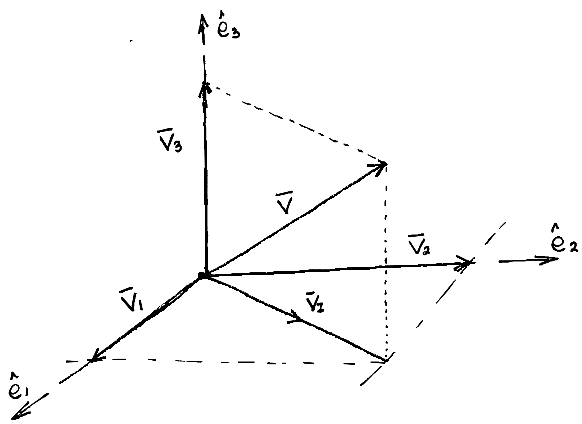


Figure 3.3: Decomposition of $\bar{V}$ into rectangular components

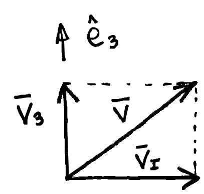


Figure 3.4: Decomposition of $\bar{V}$ into $\bar{V}_I$ and $\bar{V}_3$

We now decompose $\bar{V}_I$ into two components, $\bar{V}_1$ and $\bar{V}_2$ where $\bar{V}_1$ and $\bar{V}_2$ are parallel to $\hat{e}_1$ and $\hat{e}_2$, respectively. Thus

$$\bar{V}_I = \bar{V}_1 + \bar{V}_2 \,.$$

Also, by the Pythagorean theorem, we have

$$|\bar{V}_I|^2 = |\bar{V}_1|^2 + |\bar{V}_2|^2 \,.$$

By combining the above two decompositions, we obtain that

$$\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3$$

and

$$|\bar{V}|^2 = |\bar{V}_1|^2 + |\bar{V}_2|^2 + |\bar{V}_3|^2 \,.$$

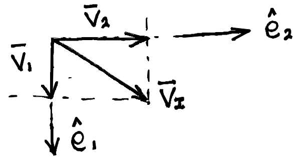


Figure 3.5: Decomposition of $\bar{V}$ into $\bar{V}_I$ and $\bar{V}_3$

Since $\bar{V}_1 = V_1\hat{e}_1$, $\bar{V}_2 = V_2\hat{e}_2$ and $\bar{V}_3 = V_3\hat{e}_3$ with $|\bar{V}_1|^2 = V_1^2$, $|\bar{V}_2|^2 = V_2^2$ and $|\bar{V}_3|^2 = V_3^2$, we obtain the desired result, namely,

$$\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3 \qquad \text{and} \qquad |\bar{V}| = (V_1^2 + V_2^2 + V_3^2)^{1/2}\,.$$

With the above decomposition, we can regard the vector $\bar{V}$ as the diagonal of a rectangular box with edges parallel to $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ and with dimensions $|\bar{V}_1|, |\bar{V}_2|$ and $|\bar{V}_3|$.

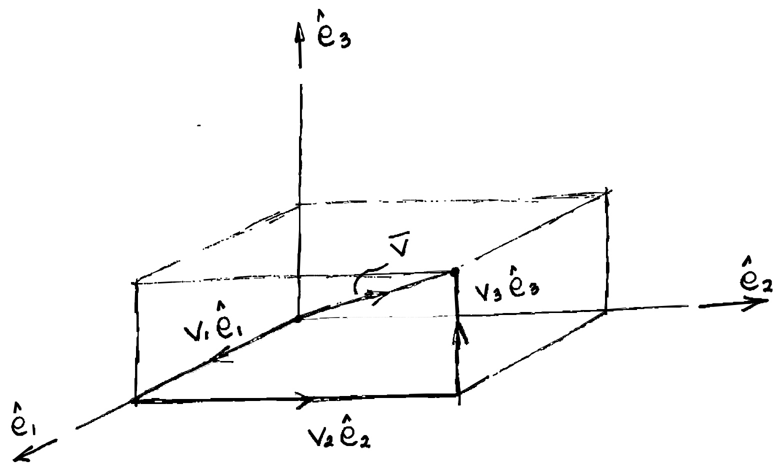


Figure 3.6: $\bar{V}$ in a box

Example 8 *Given* $\bar{V}$ and the orthogonal triad $(\bar{u}_1,\ \hat{u}_2,\ \hat{u}_3)$ as shown, *find*

(i) scalars $V_1,\ V_2,\ V_3$, such that $\bar{V} = V_1\hat{u}_1 + V_2\hat{u}_2 + V_3\hat{u}_3$

(ii) scalars $n_1,\ n_2,\ n_3$, such that $\bar{u}_{\bar{V}} = n_1\hat{u}_1 + n_2\hat{u}_2 + n_3\hat{u}_3$.

Solution:

(i) Clearly,

$$\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3$$

where $\bar{V}_1,\ \bar{V}_2,\ \bar{V}_3$ are as shown.

Also,

$$\begin{aligned}\bar{V}_1 &= |\bar{V}_1|\hat{u}_1 = 1\hat{u}_1 = \hat{u}_1 \\ \bar{V}_2 &= |\bar{V}_2|\hat{u}_2 = 2\hat{u}_2 \\ \bar{V}_3 &= |\bar{V}_3|\hat{u}_3 = 3\hat{u}_3\end{aligned}$$

Thus,

$$\bar{V} = \hat{u}_1 + 2\hat{u}_2 + 3\hat{u}_3\ ,$$

or, $\bar{V} = V_1\hat{u}_1 + V_2\hat{u}_2 + V_3\hat{u}_3$ where

$$\boxed{V_1 = 1,\ V_2 = 2,\ V_3 = 3}$$

(ii) Since

$$|\bar{V}| = \sqrt{V_1^2 + V_2^2 + V_3^2} = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}\,,$$

we have

$$\begin{aligned}\hat{u}_{\bar{V}} &= \frac{\bar{V}}{|\bar{V}|} = \frac{1}{\sqrt{14}}(\hat{u}_1 + 2u_2 + 3\hat{u}_3) = \frac{1}{\sqrt{14}}\hat{u}_1 + \frac{2}{\sqrt{14}}\hat{u}_2 + \frac{3}{\sqrt{14}}\hat{u}_3 \\ &= 0.2673\ \hat{u}_1 + 0.5345\ \hat{u}_2 + 0.8018\ \hat{u}_3\end{aligned}$$

So, $\hat{u}_{\bar{V}} = n_1\hat{u}_1 + n_2\hat{u}_2 + n_3\hat{u}_3$ where

$$\boxed{n_1 = 0.2673,\quad n_2 = 0.5345,\quad n_3 = 0.818}$$

Example 9 *Given* the vector $\bar{V}$ and the orthogonal triad $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ as shown where

$$\alpha = 63.43^\circ\,, \qquad \beta \ = \ 53.30^\circ\,, \qquad |\bar{V}| = 3.742\,,$$

find scalars $V_1,\ V_2,\ V_3$ such that $\bar{V} = V_1\hat{u}_1 + V_2\hat{u}_2 + V_3\hat{u}_3$.

Solution:

Clearly,

$$\bar{V} = \bar{V}_I + \bar{V}_3$$

with

$$|\bar{V}_I| = |\bar{V}| \cos\beta = 3.742 \cos(53.3^\circ) = 2.236$$

and

$$|\bar{V}_3| = |\bar{V}| \sin\beta = 3.742 \sin(53.3^\circ) = 3.000 \, .$$

Also,

$$\bar{V}_3 = |\bar{V}_3|\hat{u}_3 = 3\hat{u}_3$$

Considering the decomposition of $\bar{V}_I$, we have

$$\bar{V}_I = \bar{V}_1 + \bar{V}_2$$

with

$$
\begin{aligned}
|\bar{V}_1| &= |\bar{V}_I| \cos\alpha = 2.236\cos(63.43) = 1.000 \\
\bar{V}_1 &= |\bar{V}_1|\hat{u}_1 = \hat{u}_1
\end{aligned}
$$

and

$$
\begin{aligned}
|\bar{V}_2| &= |\bar{V}_I| \sin\alpha = 2.236\sin(63.43) = 2.000 \\
\bar{V}_2 &= |\bar{V}_2|\hat{u}_2 = 2\hat{u}_2
\end{aligned}
$$

Hence,

$$\bar{V} = \bar{V}_I + \bar{V}_3 = \bar{V}_1 + \bar{V}_2 + \bar{V}_3 = \hat{u}_1 + 2\hat{u}_2 + 3\hat{u}_3$$

So, $\bar{V} = V_1\hat{u}_1 + V_2\hat{u}_2 + V_3\hat{u}_3$ where

$$\boxed{V_1 = 1\,, \quad V_2 = 2\,, \quad V_3 = 3\,.}$$

Note that $\bar{V}$ is the same as the $\bar{V}$ considered in the previous example.

Example 10

Example 11

Addition of vectors via addition of scalar components

If

$$\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3 \qquad \text{and} \qquad \bar{W} = W_1\hat{e}_1 + W_2\hat{e}_2 + W_3\hat{e}_3$$

then, using the field properties of vectors, one readily obtains that

$$\boxed{\bar{V} + \bar{W} = (V_1 + W_1)\hat{e}_1 + (V_2 + W_2)\hat{e}_2 + (V_3 + W_3)\hat{e}_3}$$

Scalar multiplication of a vector via multiplication of its scalar components

If

$$\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$$

then, using the field properties of vectors, one readily obtains that

$$\boxed{k\bar{V} = (kV_1)\hat{e}_1 + (kV_2)\hat{e}_2 + (kV_3)\hat{e}_3}$$

3.5 Products of Vectors

Before discussing products of vectors, we need to examine what we mean by the angle between two vectors.

3.5.1 The angle between two vectors

Suppose $\bar{V}$ and $\bar{W}$ are two *non-zero* vectors.

We denote the angle between $\bar{V}$ and $\bar{W}$ as $\angle\bar{V}\bar{W}$.

Properties

1) $\angle\bar{W}\bar{V} = \angle\bar{V}\bar{W}$

2) If $\bar{V}$ and $\bar{W}$ have the same direction then, $\angle\bar{V}\bar{W} = 0$.

   If $\bar{W}$ is perpendicular to $\bar{V}$ then, $\angle\bar{V}\bar{W} = \frac{\pi}{2}$.

   If $\bar{W}$ is opposite in direction to $\bar{V}$ then, $\angle\bar{V}\bar{W} = \pi$.

3) In general, $0 \leq \angle\bar{V}\bar{W} \leq \pi$.

$0 < \angle \bar{V}\bar{W} < \frac{\pi}{2}$ $\qquad\qquad$ $\frac{\pi}{2} < \angle \bar{V}\bar{W} < \pi$

3.5.2 The scalar (dot) product of two vectors

Suppose $\bar{V}$ and $\bar{W}$ are any two non-zero vectors.

The scalar *(or* dot*)* product *of $\bar{V}$ and $\bar{W}$ is a scalar which is denoted by $\bar{V} \cdot \bar{W}$ and is defined by*

$$\boxed{\bar{V} \cdot \bar{W} = VW \cos\theta}$$

where V is the magnitude of $\bar{V}$, W is the magnitude of $\bar{W}$, and θ is the angle between $\bar{V}$ and $\bar{W}$.

If either $\bar{V}$ or $\bar{W}$ is the zero vector, then, $\bar{V} \cdot \bar{W}$ is defined to be zero which also equals $VW \cos\theta$ for any value of θ.

Remarks

1) Since $-1 \leq \cos\theta \leq 1$, we have

$$-VW \leq \bar{V} \cdot \bar{W} \leq VW$$

2) Suppose $\bar{V} \neq \bar{0}$ and $\bar{W} \neq \bar{0}$. Then

$$\theta = 0 \iff \bar{V} \cdot \bar{W} = VW$$

$$0 \leq \theta < \frac{\pi}{2} \iff \bar{V} \cdot \bar{W} > 0$$

$$\theta = \frac{\pi}{2} \iff \bar{V} \cdot \bar{W} = 0$$

$$\frac{\pi}{2} < \theta \le \pi \iff \bar{V}\cdot\bar{W} < 0$$

$$\theta = \pi \iff \bar{V}\cdot\bar{W} = -VW$$

3) Since the angle between $\bar{V}$ and itself is zero, it follows that $\bar{V}\cdot\bar{V} = V^2$; hence the magnitude of $\bar{V}$ can be expressed as

$$V = (\bar{V}\cdot\bar{V})^{1/2}\,.$$

4) If $\bar{V}$ and $\bar{W}$ represent physical quantities,

$$\dim[\bar{V}\cdot\bar{W}] = \dim[\bar{V}]\dim[\bar{W}]$$

Basic Properties of the scalar product

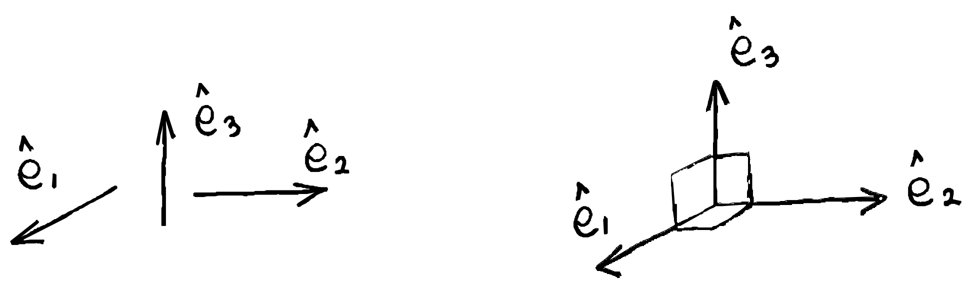
1) $\bar{V}\cdot\bar{W} = \bar{W}\cdot\bar{V}$ (commutativity)

2) $\bar{U}\cdot(\bar{V}+\bar{W}) = \bar{U}\cdot\bar{V} + \bar{U}\cdot\bar{W}$

3) $\bar{V}\cdot(k\bar{W}) = k(\bar{V}\cdot\bar{W})$

The scalar product and rectangular components

Suppose $(\hat{e}_1,\,\hat{e}_2,\,\hat{e}_3)$ is any orthogonal triad.



Facts

(1)

$$\hat{e}_i\cdot\hat{e}_j = \left\{\begin{array}{ll} 1 & i=j \\ 0 & i\neq j \end{array}\right\} \triangleq \delta_{ij}$$

For example,

$$\begin{aligned}\hat{e}_1 \cdot \hat{e}_1 &= |\hat{e}_1|^2 = 1 \\ \hat{e}_1 \cdot \hat{e}_2 &= |\hat{e}_1|\ |\hat{e}_2| \cos\left(\frac{\pi}{2}\right) = 0\end{aligned}$$

(2) If $\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$, then

$$\boxed{V_1 = \bar{V} \cdot \hat{e}_1\,, \qquad V_2 = \bar{V} \cdot \hat{e}_2\,, \qquad V_3 = \bar{V} \cdot \hat{e}_3}$$

Proof. Consider V_1. Using the properties of the dot product, we obtain that

$$\begin{aligned}\bar{V} \cdot \hat{e}_1 &= (V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3) \cdot \hat{e}_1 \\ &= V_1(\hat{e}_1 \cdot \hat{e}_1) + V_2(\hat{e}_2 \cdot \hat{e}_1) + V_3(\hat{e}_3 \cdot \hat{e}_1) \\ &= V_1(1) + V_2(0) + V_3(0) = V_1\,.\end{aligned}$$

Similarly for V_2 and V_3.

(3) If

$$\bar{V} = \bar{V}_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3 \qquad \text{and} \qquad \bar{W} = W_1\hat{e}_1 + W_2\hat{e}_2 + W_3\hat{e}_3\ ,$$

then

$$\boxed{\bar{V} \cdot \bar{W} = V_1W_1 + V_2W_2 + V_3W_3}$$

Proof. Using the properties of the dot product and the previous result, we obtain that

$$\begin{aligned}\bar{V} \cdot \bar{W} &= \bar{V} \cdot (W_1\hat{e}_1 + W_2\hat{e}_2 + W_3\hat{e}_3) \\ &= W_1(\bar{V} \cdot \hat{e}_1) + W_2(\bar{V} \cdot \hat{e}_2) + W_3(\bar{V} \cdot \hat{e}_3) \\ &= W_1V_1 + W_2V_2 + W_3V_3\,.\end{aligned}$$

(4) If θ is the angle between $\bar{V}$ and $\bar{W}$ then,

$$\boxed{\cos\theta = \frac{V_1W_1 + V_2W_2 + V_3W_3}{VW}}$$

where $V = |\bar{V}|$ and $W = |\bar{W}|$.
Proof. Recall that

$$VW\cos\theta = \bar{V} \cdot \bar{W} = V_1W_1 + V_2W_2 + V_3W_3\,.$$

Hence,

$$\cos\theta = \frac{V_1W_1 + V_2W_2 + V_3W_3}{VW}\,.$$

Also note that

$$\boxed{\theta = \cos^{-1}\left(\frac{V_1W_1 + V_2W_2 + V_3W_3}{VW}\right)}$$

Example 12 Suppose

$$\bar{V} = \hat{e}_1 + \hat{e}_2 + \hat{e}_3 \qquad \text{and} \qquad \bar{W} = \hat{e}_1 - \hat{e}_3\,.$$

Then

$$\bar{V} \cdot \bar{W} = (1)(1) + (1)(0) + (1)(-1) = 0.$$

Since $\bar{V} \cdot \bar{W}$ is zero, $\bar{V}$ is perpendicular to $\bar{W}$.

Example 13

3.5.3 Cross (vector) product of two vectors

Suppose $\bar{V}$ and $\bar{W}$ are any two non-zero non-parallel vectors.

The cross *(or* vector*)* product *of $\bar{V}$ and $\bar{W}$ is a vector which is denoted by $\bar{V} \times \bar{W}$ and is defined by*

$$\boxed{\bar{V} \times \bar{W} = VW \sin\theta \, \hat{n}}$$

where V is the magnitude of $\bar{V}$, W is the magnitude of $\bar{W}$, θ is the angle between $\bar{V}$ and $\bar{W}$ and $\hat{n}$ is the unit vector which is normal (perpendicular) to both $\bar{V}$ and $\bar{W}$ and whose direction is given by the right-hand rule*.*

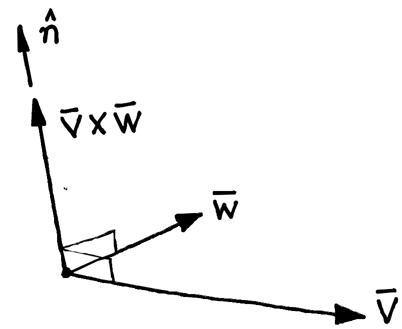


Figure 3.7: Cross product

If $\bar{V}$ and $\bar{W}$ are parallel or either of them equals $\bar{0}$ then, $\bar{V} \times \bar{W}$ is defined to be the zero vector.

Remarks

(1) Since $0 \leq \sin\theta \leq 1$ for $0 \leq \theta \leq \pi$, it follows that

$$|\bar{V} \times \bar{W}| = VW \sin\theta$$

and

$$|\bar{V} \times \bar{W}| \leq VW\,.$$

Figure 3.8: Sine function

(2) Suppose $\bar{V}$ and $\bar{W}$ are both nonzero. Then the following relationships hold.

$$\bar{V} \times \bar{W} = \bar{0} \qquad \Longleftrightarrow \qquad \bar{V} \text{ is parallel to } \bar{W}$$

$$|\bar{V} \times \bar{W}| = VW \qquad \Longleftrightarrow \qquad \bar{V} \text{ is perpendicular to } \bar{W}$$

(3) If $\bar{V}$ and $\bar{W}$ represent physical quantities, then

$$\dim\ [\bar{V} \times \bar{W}] = \dim[\bar{V}]\,\dim[\bar{W}]\ .$$

Basic Properties of the cross product.

1) $\bar{W} \times \bar{V} = -\bar{V} \times \bar{W}$ (not commutative)

2) $\bar{U} \times (\bar{V} + \bar{W}) = (\bar{U} \times \bar{V}) + (\bar{U} \times \bar{W})$ and $(\bar{U} + \bar{V}) \times \bar{W} = (\bar{U} \times \bar{V}) + (\bar{U} \times \bar{W})$

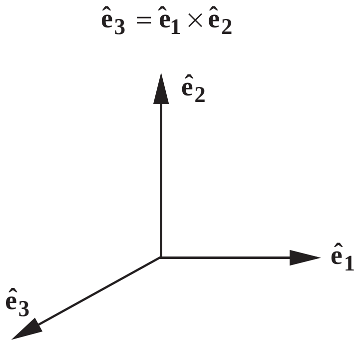
3) $\bar{V} \times (k\bar{W}) = (k\bar{V}) \times \bar{W} = k(\bar{V} \times \bar{W})$

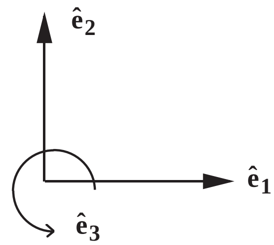
Cross product and rectangular components

An orthogonal triad, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, is said to be right-handed *if*

$$\boxed{\hat{e}_3 = \hat{e}_1 \times \hat{e}_2}$$

From now on, we shall consider only right-handed orthogonal triads.

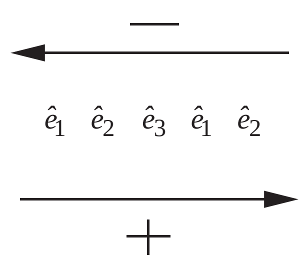




Facts
(1)

$$\begin{array}{lll} \hat{e}_1 \times \hat{e}_1 = \bar{0} & \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 & \hat{e}_1 \times \hat{e}_3 = -\hat{e}_2 \\ \hat{e}_2 \times \hat{e}_1 = -\hat{e}_3 & \hat{e}_2 \times \hat{e}_2 = \bar{0} & \hat{e}_2 \times \hat{e}_3 = \hat{e}_1 \\ \hat{e}_3 \times \hat{e}_1 = \hat{e}_2 & \hat{e}_3 \times \hat{e}_2 = -\hat{e}_1 & \hat{e}_3 \times \hat{e}_3 = \bar{0} \end{array}$$

These relationships are illustrated below.



(2) If

$$\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3 \qquad \text{and} \qquad \bar{W} = W_1\hat{e}_1 + W_2\hat{e}_2 + W_3\hat{e}_3\,,$$

then

$$\boxed{\bar{V} \times \bar{W} = (V_2W_3 - V_3W_2)\ \hat{e}_1 + (V_3W_1 - V_1W_3)\ \hat{e}_2 + (V_1W_2 - V_2W_1)\ \hat{e}_3}$$

Proof. Exercise

(3) the above expression for $\bar{V} \times \bar{W}$ may also be obtained from

$$\bar{V} \times \bar{W} = \det \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ V_1 & V_2 & V_3 \\ W_2 & W_2 & W_3 \end{pmatrix}$$

where det denotes determinant.
Proof. Exercise

3.5.4 Triple products

Scalar triple product

The scalar triple product *of three vectors $\bar{U}$, $\bar{V}$ and $\bar{W}$ is the scalar defined by*

$$\boxed{\bar{U}\cdot(\bar{V}\times\bar{W})}$$

Facts

(1)

$$\bar{U}\cdot(\bar{V}\times\bar{W}) = \det\begin{pmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{pmatrix}$$

(2)

$$\boxed{\bar{U}\cdot(\bar{V}\times\bar{W}) = (\bar{U}\times\bar{V})\cdot\bar{W}}$$

that is, $\cdot$ and $\times$ can be interchanged.
Proof. Exercise

Vector triple product

The vector triple product *of three vectors $\bar{U}$, $\bar{V}$ and $\bar{W}$ is the vector defined by*

$$\boxed{\bar{U}\times(\bar{V}\times\bar{W})}$$

Facts

(1) In general,

$$\bar{U}\times(\bar{V}\times\bar{W}) \neq (\bar{U}\times\bar{V})\times\bar{W}\,.$$

For example,

$$\begin{aligned} \hat{e}_1\times(\hat{e}_1\times\hat{e}_2) &= \hat{e}_1\times\hat{e}_3 = -\hat{e}_2 \\ (\hat{e}_1\times\hat{e}_1)\times\hat{e}_2 &= \bar{0}\times\hat{e}_2 = \bar{0} \end{aligned}$$

(2)

$$\boxed{\bar{U}\times(\bar{V}\times\bar{W}) = (\bar{U}\cdot\bar{W})\bar{V} - (\bar{U}\cdot\bar{V})\bar{W}}$$

Proof. Exercise

Chapter 4

Kinematics of Points

In kinematics, we are concerned with motion without being concerned about what causes the motion. If a body is small in comparison to its "surroundings", we can view the body as occupying a single point at each instant of time. We will also be interested in the motion of points on "large" bodies. The kinematics of points involves the concepts of time, position, velocity and acceleration.

4.1 Derivatives

To involve ourselves with kinematics, we need derivatives.

4.1.1 Scalar functions

First, consider the situation where v is a scalar function of a scalar variable t. Suppose t_1 is a specific value of t. Then the formal definition of the derivative of v at t_1 is

$$\frac{dv}{dt}(t_1) = \lim_{t \to t_1} \frac{v(t) - v(t_1)}{t - t_1}$$

Sometimes this is called the first derivative of v. Oftentimes, $\frac{dv}{dt}$ is denoted by $\dot{v}$. A graphical representation is given in Figure 4.1.

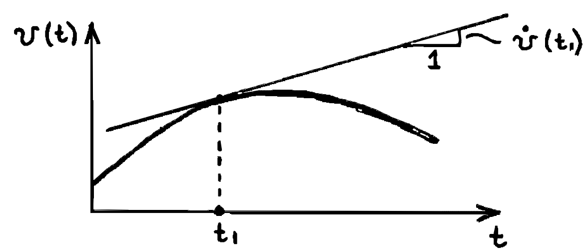


Figure 4.1: Derivative of a scalar function

The second derivative of v, denoted by $\frac{d^2v}{dt^2}$ or $\ddot{v}$, is defined as the derivative of the first derivative of v, that is

$$\frac{d^2v}{dt^2} = \frac{d}{dt}\left(\frac{dv}{dt}\right) .$$

In evaluating derivatives, we normally do not have to resort to the above definition. By knowing the derivatives of commonly used functions (such as cos, sin, and polynomials) and using the following properties, one can usually compute the derivatives of most commonly encountered functions.

Properties. The following hold for any two scalar functions v and w.

(a)

$$\frac{d}{dt}(v+w) = \frac{dv}{dt} + \frac{dw}{dt}$$

(b) (Product rule)

$$\frac{d}{dt}(vw) = \frac{dv}{dt}w + v\frac{dw}{dt}$$

(c) (Quotient rule) Whenever $w(t) \neq 0$,

$$\frac{d}{dt}\left(\frac{v}{w}\right) = \frac{\frac{dv}{dt}w - v\frac{dw}{dt}}{w^2}$$

(d) (Chain rule)

$$\frac{d}{dt}(v(w)) = \frac{dv}{dw}\frac{dw}{dt}$$

Example 14 Consider the function given by $f(t) = \cos(t^2)$. Then

$$f(t) = v(w(t)) \qquad \text{where} \qquad v(w) = \cos w \quad \text{and} \quad w(t) = t^2\,.$$

Applying the chain rule, we obtain that

$$\dot{f} = \frac{df}{dt} = \frac{dv}{dw}\frac{dw}{dt} = (-\sin w)(2t)\,.$$

Hence,

$$\dot{f} = -2t\sin(t^2)\,.$$

Exercises

Exercise 7 Compute the first and second derivatives of the following functions.

(a) $\theta(t) = \cos(20t)$

(b) $f(t) = e^{t^2}$

(c) $x(t) = \sin(e^t)$

(d) $h(t) = e^{2t}\cos(10t)$

Exercise 8 Compute the derivative of the following functions.

(a) $y(t) = \dfrac{\sin(10t)}{1+t^2}$

(b) $z(t) = t^2 e^{3t}\sin(4t)$

4.1.2 Vector functions

Consider now the situation where $\bar{V}$ is a *vector* function of a scalar variable t. Suppose t_1 is a specific value of t. Then the formal definition of the derivative of $\bar{V}$ at t_1 is

$$\frac{d\bar{V}}{dt}(t_1) = \lim_{t\to t_1} \frac{\bar{V}(t) - \bar{V}(t_1)}{t - t_1}$$

Oftentimes, $\frac{d\bar{V}}{dt}$ is denoted by $\dot{\bar{V}}$.

Properties. The following hold for any two vector functions $\bar{V}$ and $\bar{W}$ and any scalar function k.

(a)

$$\frac{d}{dt}(\bar{V} + \bar{W}) = \frac{d\bar{V}}{dt} + \frac{d\bar{W}}{dt}$$

(b)

$$\frac{d}{dt}(k\bar{V}) = \frac{dk}{dt}\bar{V} + k\frac{d\bar{V}}{dt}$$

(c)

$$\frac{d}{dt}(\bar{V} \cdot \bar{W}) = \frac{d\bar{V}}{dt} \cdot \bar{W} + \bar{V} \cdot \frac{d\bar{W}}{dt}$$

(d)

$$\frac{d}{dt}(\bar{V} \times \bar{W}) = \frac{d\bar{V}}{dt} \times \bar{W} + \bar{V} \times \frac{d\bar{W}}{dt}$$

(e) (Chain rule)

$$\frac{d}{dt}(\bar{V}(k)) = \frac{dk}{dt}\,\frac{d\bar{V}}{dk}$$

Derivatives and components. Usually we evaluate the derivative of a vector function by differentiating its scalar components. Suppose $\hat{e}_1, \hat{e}_2, \hat{e}_3$ is a set of constant basis vectors and

$$\bar{V}(t) = V_1(t)\hat{e}_1 + V_2(t)\hat{e}_2 + V_3(t)\hat{e}_3\,.$$

Then, using properties (a) and (b) above, we obtain that

$$\frac{d\bar{V}}{dt} = \frac{dV_1}{dt}\hat{e}_1 + \frac{dV_2}{dt}\hat{e}_2 + \frac{dV_3}{dt}\hat{e}_3$$

or

$$\dot{\bar{V}} = \dot{V}_1\hat{e}_1 + \dot{V}_2\hat{e}_2 + \dot{V}_3\hat{e}_3\,.$$

4.1.3 The frame derivative of a vector function

We define a reference frame (or frame of reference) to be an right-handed orthogonal triad of unit vectors which have the same point of application. Figure 4.2 illustrates several reference frames. Usually we use a single symbol to reference frame; thus the reference frame consisting

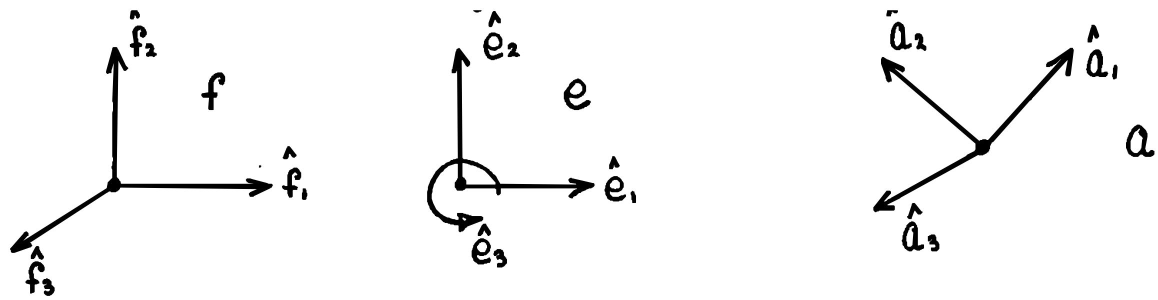


Figure 4.2: Reference frames

of the vectors $\hat{f}_1, \hat{f}_2, \hat{f}_3$ will be referred to as the reference frame f.

Consider a time-varying vector $\bar{V}$. If one observes this vector from different reference frames then, one will observe different variations of $\bar{V}$ with time. For that reason, we have the following definition.

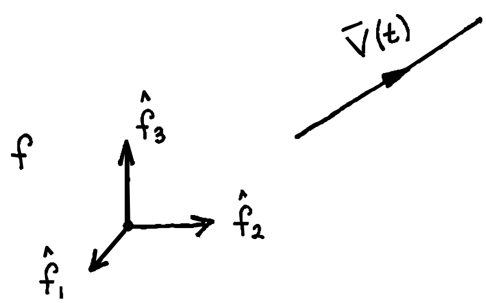


Figure 4.3: The frame derivative of a vector

The derivative of $\bar{V}$ in f is denoted by $\frac{{}^f d\bar{V}}{dt}$ and is defined by

$$\boxed{\frac{{}^f d\bar{V}}{dt} = \frac{dV_1}{dt}\hat{f}_1 + \frac{dV_2}{dt}\hat{f}_2 + \frac{dV_3}{dt}\hat{f}_3}$$

where V_1, V_2, V_3 are the scalar components of $\bar{V}$ relative to f, that is,

$$\bar{V} = V_1\hat{f}_1 + V_2\hat{f}_2 + V_3\hat{f}_3 \,.$$

Oftentimes, $\frac{{}^f d\bar{V}}{dt}$ is denoted by ${}^f\dot{\bar{V}}$. Thus,

$${}^f\dot{\bar{V}} = \dot{V}_1\hat{f}_1 + \dot{V}_2\hat{f}_2 + \dot{V}_3\hat{f}_3 \,.$$

Properties. The following hold for any two vector functions $\bar{V}$ and $\bar{W}$ and any scalar function k.

(a)

$$\frac{{}^f d}{dt}(\bar{V} + \bar{W}) = \frac{{}^f d\bar{V}}{dt} + \frac{{}^f d\bar{W}}{dt}$$

(b)

$$\frac{{}^f d}{dt}(k\bar{V}) = \frac{dk}{dt}\bar{V} + k\frac{{}^f d\bar{V}}{dt}$$

(c)

$$\frac{{}^f d}{dt}(\bar{V} \cdot \bar{W}) = \frac{{}^f d\bar{V}}{dt} \cdot \bar{W} + \bar{V} \cdot \frac{{}^f d\bar{W}}{dt}$$

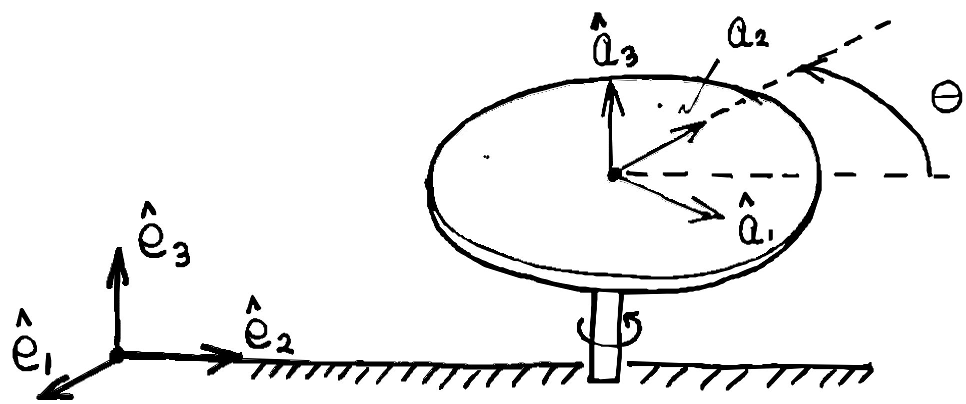
(d)

$$\frac{{}^f d}{dt}(\bar{V} \times \bar{W}) = \frac{{}^f d\bar{V}}{dt} \times \bar{W} + \bar{V} \times \frac{{}^f d\bar{W}}{dt}$$

(e) (Chain rule)

$$\frac{{}^f d}{dt}(\bar{V}(k)) = \frac{dk}{dt}\,\frac{{}^f d\bar{V}}{dk}$$

Example 15



Exercises

Exercise 9 Consider the reference frames $f = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ and $g = (\hat{g}_1, \hat{g}_2, \hat{g}_3)$ as illustrated where $\theta = 2t$ rads. Suppose the vector $\bar{Z}$ is given by

$$\bar{Z} = 2t\hat{g}_1 + t^2\hat{g}_2 \,.$$

In terms of t and the units vectors of g, find expressions for the following quantities.

(a) ${}^g\dot{\bar{Z}}$

(b) ${}^f\dot{\bar{Z}}$

(c) ${}^g\bar{Z} + \bar{\omega} \times \bar{Z}$ where $\bar{\omega} = \dot{\theta}\hat{g}_3$

Compare the answers for parts (b) and (c).

4.2 Basic definitions

Besides time, there are three additional basic concepts in the kinematics of points, namely, position, velocity, and acceleration.

4.2.1 Position

Consider any two points O and P. We define the position of P relative to O (denoted $\overline{r}^{OP}$) or the position vector from O to P as the vector from O to P, that is,

$$\boxed{\overline{r}^{OP} := \overline{OP}}$$

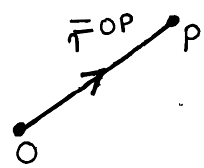


Figure 4.4: Position vector, $\overline{r}^{OP}$

Clearly, the position of a point O relative to itself is the zero vector, that is,

$$\overline{r}^{OO} = \bar{0}\,.$$

It should also be clear that the position of O relative to P is the negative of the position of P relative to O, that is,

$$\overline{r}^{PO} = -\overline{r}^{OP}\,.$$

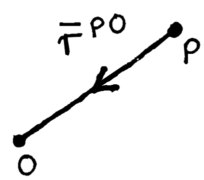


Figure 4.5: $\overline{r}^{PO}$

Composition of position vectors. For any three points O, P, Q, we have

$$\overline{r}^{OQ} = \overline{r}^{OP} + \overline{r}^{PQ} \tag{4.1}$$

This follows from the triangle law of vector addition and is illustrated in Figure 4.6.

From the above relationship, we also have

$$\overline{r}^{PQ} = \overline{r}^{OQ} - \overline{r}^{OP}\,.$$

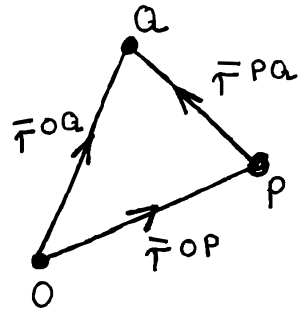


Figure 4.6: Composition of position vectors

Consider now several points, $P_1, P_2, \ldots, P_n$. Then, by repeated application of result (4.1), we obtain

$$\overline{r}^{P_1 P_n} = \overline{r}^{P_1 P_2} + \overline{r}^{P_2 P_3} + \ldots + \overline{r}^{P_{n-1} P_n} .$$

This is illustrated in Figure 4.7.

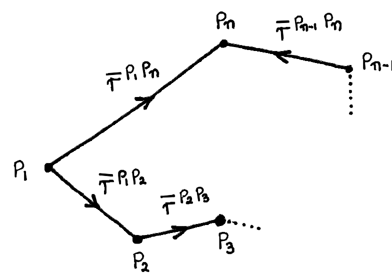


Figure 4.7: Composition of several position vectors

4.2.2 Velocity and acceleration

Suppose we are observing the motion of some point P from a reference frame f. We first demonstrate the following result.

If O and O' are any two points which are fixed in reference frame f, then

$$\frac{{}^f d}{dt} \overline{r}^{OP} = \frac{{}^f d}{dt} \overline{r}^{O'P}$$

To see this, first note that

$$\overline{r}^{OP} = \overline{r}^{OO'} + \overline{r}^{O'P} .$$

Hence,

$$\frac{{}^f d}{dt} \overline{r}^{OP} = \frac{{}^f d}{dt} \overline{r}^{OO'} + \frac{{}^f d}{dt} \overline{r}^{O'P}$$

Since points O and O' are fixed in f, the vector $\overline{r}^{OO'}$ is a fixed vector in f, hence

$$\frac{{}^f d}{dt} \overline{r}^{OO'} = \overline{0}$$

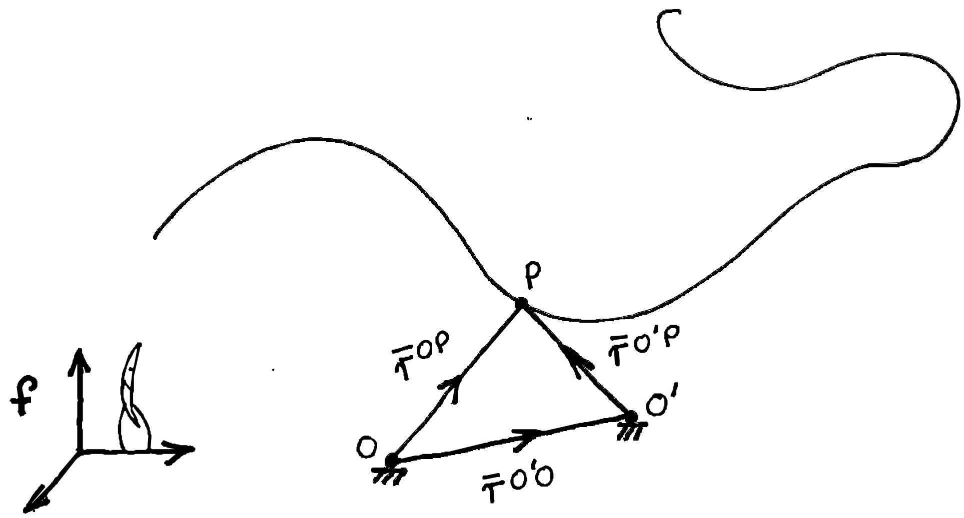


Figure 4.8: Independence of velocity on origin

and the desired result follows. ■

We define the velocity of P in f (denoted ${}^f\overline{v}^P$) by

$$\boxed{{}^f\overline{v}^P := \frac{{}^fd}{dt}\overline{r}^{OP}}$$

where O is any point fixed in f. The speed of P in f is $|{}^f\overline{v}^P|$, the magnitude of the velocity of P in f.

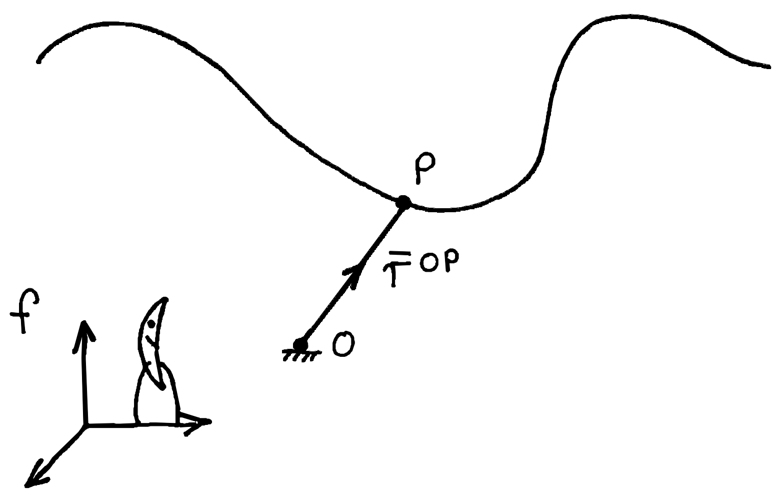


Figure 4.9: The velocity of P in f

We define the acceleration of P in f (denoted ${}^f\overline{a}^P$) by

$$\boxed{{}^f\overline{a}^P := \frac{{}^fd}{dt}\,{}^f\overline{v}^P}$$

Note that

$$^f\overline{a}^P = \frac{^f d^2}{dt^2}\,\overline{r}^{OP}$$

In the next section, we consider some special types of motions. First we have some examples to illustrate the above concepts.

Example 16 (Pendulum with moving support)

Example 17 (Bug on bar on cart)

Exercises

Exercise 10 The two link planar manipulator is constrained to move in the plane defined by the vectors $\hat{e}_1$ and $\hat{e}_2$ of reference frame e. Point O is fixed in e.

Find expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$ in terms of θ_1, θ_2, their first and second time derivatives, l_1, l_2 and $\hat{e}_1, \hat{e}_2, \hat{e}_3$.

Exercise 11 The small ball P moves in the straight slot which is fixed in the disk. Relative to reference frame e, point O is fixed and the disk rotates about an axis through O which is parallel to $\hat{e}_3$ and perpendicular to the plane of the disk.

Find expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$ in terms of l, r, θ, $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, $\ddot{\theta}$ and unit vectors fixed in the disk.

4.3 Rectilinear motion

The simplest type of motion is rectilinear. The motion of point P in a reference frame f is called rectilinear if P always moves in a straight line fixed in f. If we choose a reference

Figure 4.10: Rectilinear motion

point O which is along the line and fixed in f and if we choose one direction along the line as a positive direction, then the location of P can be uniquely specified by specifying the displacement x of P from O. Thus we can describe rectilinear motion with a single scalar. In Figure 4.11, the displacement x is considered positive when P is to the right of O.

Figure 4.11: Displacement x

Since we are only dealing with the motion of one point P relative to a single reference frame f, we simplify notation here and let $\bar{r}$ be the position of P relative to O, $\bar{v}$ be the velocity of P in f and $\bar{a}$ be the acceleration of P in f.

Figure 4.12: $\hat{e}$

If we introduce a unit vector $\hat{e}$ which along the line of motion and pointing in the positive direction for displacement along the line, then $\bar{r} = x\hat{e}$. Since O is fixed in f, we have

$$\bar{v} = \frac{{}^f d\bar{r}}{dt} = \frac{{}^f d}{dt}(x\hat{e}) = \dot{x}\hat{e}\,.$$

The last equality above follows from the fact that $\hat{e}$ is a constant vector in reference frame f. We also have that

$$\bar{a} = \frac{{}^f d\bar{v}}{dt} = \frac{{}^f d}{dt}(\dot{x}\hat{e}) = \ddot{x}\hat{e}\,.$$

So, summarizing, we have

$$\bar{r} = x\hat{e}\,, \qquad \bar{v} = v\hat{e}\,, \qquad \bar{a} = a\hat{e}$$

where

$$v = \dot{x} \qquad \text{and} \qquad a = \dot{v} = \ddot{x}\,. \tag{4.2}$$

4.4 Planar motion

The motion of point P in a reference frame f is called planar if P always moves in a plane which is fixed in f.

Figure 4.13: Planar motion

If we choose a reference point O which is in the plane and fixed in f, then the location of P can be uniquely specified by specifying the position of P relative to O. For the rest of this section, we let $\bar{r}$ be the position of P relative to O, $\bar{v}$ be the velocity of P in f and $\bar{a}$ be the acceleration of P in f. Hence

$$\bar{v} = \frac{d\bar{r}}{dt} \qquad \text{and} \qquad \bar{a} = \frac{d\bar{v}}{dt}$$

where it is understood that the above differentiations are carried out relative to frame f.

In general, planar motion can be described with two scalar coordinates. We now consider two different coordinate systems for describing planar motion, namely cartesian coordinates and polar coordinates.

4.4.1 Cartesian coordinates

Choose any two mutually perpendicular lines in the plane passing through O and fixed in f. By choosing a positive direction for each line, The location of point P can be uniquely determined by the cartesian coordinates x and y as illustrated in Figure 4.14.

Figure 4.14: Cartesian coordinates x and y

Figure 4.15: $\hat{e}_1$ and $\hat{e}_2$

We now compute expressions for $\bar{r}$, $\bar{v}$ and $\bar{a}$ in terms of cartesian coordinates. To this end, we introduce unit vectors $\hat{e}_1$, $\hat{e}_2$ fixed in the plane as illustrated in Figure 4.15. Then, the position of P relative to O can be expressed as

$$\bar{r} = x\hat{e}_1 + y\hat{e}_2$$

Since $\hat{e}_1$ and $\hat{e}_2$ are fixed vectors in f, differentiating the above expression in f yields the the velocity of P in f :

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(x\hat{e}_1 + y\hat{e}_2) = \dot{x}\hat{e}_1 + \dot{y}\hat{e}_2\,.$$

Differentiating once more yields the acceleration of P in f :

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(\dot{x}\hat{e}_1 + \dot{y}\hat{e}_2) = \ddot{x}\hat{e}_1 + \ddot{y}\hat{e}_2\,.$$

So, summarizing, we have

$$\boxed{\begin{array}{lllll} \bar{r} = x\hat{e}_1 + y\hat{e}_2 & & & & \\ \bar{v} = v_1\hat{e}_1 + v_2\hat{e}_2 & \text{where} & v_1 = \dot{x} & \text{and} & v_2 = \dot{y} \\ \bar{a} = a_1\hat{e}_1 + a_2\hat{e}_2 & \text{where} & a_1 = \dot{v}_1 = \ddot{x} & \text{and} & a_2 = \dot{v}_2 = \ddot{y} \end{array}} \qquad (4.3)$$

4.4.2 Projectiles

As an application of cartesian coordinates, let us consider the motion of a projectile near the surface of YFHB (your favorite heavenly body, for example, the earth or the dark side of the moon). Suppose that a projectile P is launched from point O on YFHB at a launch

Figure 4.16: Projectile motion

angle θ and with launch speed v relative to YFHB. Modeling YFHB as flat and neglecting all forces other than gravitational forces, then relative to YFHB, P move in a vertical plane and its acceleration is given by

$$\bar{a} = g\hat{g}$$

where $\hat{g}$ is the unit vector in the direction of the local vertical and g is the gravitational acceleration of YFHB. We will show this fact later in the course.

Introduce reference frame e fixed in YFHB with origin at O as shown. Then, the position of P is completely described by the cartesian coordinates x, y where y is the height of P above the surface of YFHB and we call x the horizontal range. Let $\bar{v}$ and $\bar{a}$ be the velocity and acceleration, respectively, of P in e. Then,

$$\bar{a} = -g\hat{e}_2\,.$$

Hence, it follows from (4.3) that

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g\,. \tag{4.4}$$

Choosing t to be zero when P is launched from O, the velocity of P at launch, that is $\bar{v}(0)$, is given by

$$\bar{v}(0) = v\cos\theta\,\hat{e}_1 + v\sin\theta\,\hat{e}_2\,.$$

Hence, it follows from (4.3) that

$$\dot{x}(0) = v\cos\theta \qquad \text{and} \qquad \dot{y}(0) = v\sin\theta\,. \tag{4.5}$$

Integrating relationships (4.4) from 0 to t and using the initial conditions (4.5), we obtain that

$$\dot{x}(t) = v\cos\theta \qquad \text{and} \qquad \dot{y}(t) = v\sin\theta - gt\,. \tag{4.6}$$

Since $x(0) = 0$ and $y(0) = 0$ we can integrate (4.6) from 0 to t to obtain

$$\boxed{x(t) = (v\cos\theta)t \qquad \text{and} \qquad y(t) = (v\sin\theta)t - \tfrac{1}{2}gt^2\,.} \tag{4.7}$$

Note that if we use the first equation above to express t in terms of x and then substitute this expression for t into the second equation, we obtain

$$y = (\tan\theta)x - \left(\frac{g}{2v^2\cos\theta^2}\right)x^2\,. \tag{4.8}$$

This equation tells us that the trajectory of the projectile is parabolic.

Maximum height. If t_h is the time at which P reaches its maximum height h, we must have $\dot{y}(t_h) = 0$. Hence, using (4.6), we obtain that

$$\dot{y}(t_h) = v\sin\theta - gt_h = 0\,.$$

Solving for t_h yields

$$t_h = \frac{v\sin\theta}{g}\,.$$

Since $h = y(t_h)$, substitution for t_h into the second equation in (4.7) yields

$$\boxed{h = \frac{v^2\sin^2\theta}{2g}} \tag{4.9}$$

Figure 4.17: Maximum projectile height

Range at impact. Letting l be the horizontal range when P impacts YFHB and t_l the corresponding time, we have $y(t_l) = 0$. Hence

Figure 4.18: Range at impact

$$y(t_l) = (v \sin\theta)t_l - \frac{1}{2}gt_l^2 = 0\,.$$

This last equation has two solutions for t_l, namely $t_l = 0$ and

$$t_l = \frac{2v\sin\theta}{g}\,.$$

It is the second solution we want. Note that this is twice the time that the projectile took to reach maximum height. We now obtain that

$$l = x(t_l) = \frac{2v^2\sin\theta\cos\theta}{g}\,.$$

Noting that $2\sin\theta\cos\theta = \sin(2\theta)$ we have

$$\boxed{l = \frac{v^2\sin(2\theta)}{g}} \tag{4.10}$$

It should be clear from the last expression, that if one wants to maximize the range of the projectile for a given launch speed, then one must choose the launch angle θ to be $45°$.

4.4.3 Polar coordinates

There are some situations in which it is more convenient to use polar coordinates instead of cartesian coordinates to describe planar motion. We shall see this later when we look at the motion of a satellite in orbit about YFHB (your favorite heavenly body). To describe the position of point P relative to O, we first introduce a half-line which is fixed in reference

frame f, lies in the plane of motion of P and which starts at O. Then, the polar coordinates which describe the position of P are (r, θ) where r is the distance between O and P and θ is the angle between the line segment OP and the chosen reference half-line; θ is considered positive when counterclockwise. We now compute expressions for $\bar{r}$, $\bar{v}$ and $\bar{a}$ in terms of polar coordinates

Figure 4.19: Polar coordinates

Figure 4.20: $\hat{e}_1$ and $\hat{e}_2$

Introduce unit vectors $\hat{e}_1$, $\hat{e}_2$ fixed in the plane as illustrated in Figure 4.20. Then, the position of P relative to O can be expressed as

$$\bar{r} = rC_\theta \hat{e}_1 + rS_\theta \hat{e}_2 \,.$$

Since $\hat{e}_1$ and $\hat{e}_2$ are fixed vectors (in f), differentiating the above expression (in f) yields the the velocity of P (in f):

$$\bar{v} = (\dot{r}C_\theta - r\dot{\theta}S_\theta)\hat{e}_1 + (\dot{r}S_\theta + r\dot{\theta}C_\theta)\hat{e}_2 \,.$$

Differentiating once more (groan!) yields the acceleration of P (in f):

$$\bar{a} = (\ddot{r}C_\theta - 2\dot{r}\dot{\theta}S_\theta - r\ddot{\theta}S_\theta - r\dot{\theta}^2 C_\theta)\hat{e}_1 + (\ddot{r}S_\theta + 2\dot{r}\dot{\theta}C_\theta + r\ddot{\theta}C_\theta - r\dot{\theta}^2 S_\theta)\hat{e}_2$$

To obtain much simpler expressions for $\bar{r}$, $\bar{v}$, and $\bar{a}$, we introduce two new unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ as illustrated in Figure 4.21 Considering the relationships between $\hat{e}_r, \hat{e}_\theta$ and $\hat{e}_1, \hat{e}_2$ (see Figure 4.22) we have

$$\hat{e}_r = C_\theta \hat{e}_1 + S_\theta \hat{e}_2 \quad \text{and} \quad \hat{e}_\theta = -S_\theta \hat{e}_1 + C_\theta \hat{e}_2 \,.$$

Recalling our expression for $\bar{r}$, we obtain that

$$\begin{aligned} \bar{r} &= r(C_\theta \hat{e}_1 + S_\theta \hat{e}_2) \\ &= r\hat{e}_r \,. \end{aligned}$$

Figure 4.21: $\hat{e}_r$ and $\hat{e}_\theta$

Figure 4.22:

This is as expected since, r is the magnitude of r and $\hat{e}_r$ is the unit vector in the direction of $\bar{r}$.

Rearranging our expression for $\bar{v}$, we see that

$$\begin{aligned} \bar{v} &= \dot{r}(C_\theta\hat{e}_1 + S_\theta\hat{e}_2) + r\dot{\theta}(-S_\theta\hat{e}_1 + C_\theta\hat{e}_2) \\ &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\,. \end{aligned}$$

This is much simpler!

Rearranging our expression for $\bar{a}$, we see that

$$\begin{aligned} \bar{a} &= (\ddot{r} - r\dot{\theta}^2)(C_\theta\hat{e}_1 + S_\theta\hat{e}_2) + (r\ddot{\theta} + 2\dot{r}\dot{\theta})(-S_\theta\hat{e}_1 + C_\theta\hat{e}_2) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\,. \end{aligned}$$

Much simpler!

Summarizing, we obtain the following expressions for position, velocity, and acceleration in terms of polar coordinates.

$$\boxed{\begin{array}{lllll} \bar{r} = r\hat{e}_r & & & & \\ \bar{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta & \text{where} & v_r = \dot{r} & \text{and} & v_\theta = r\dot{\theta} \\ \bar{a} = a_r\hat{e}_r + a_\theta\hat{e}_\theta & \text{where} & a_r = \ddot{r} - r\dot{\theta}^2 & \text{and} & a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{array}}$$

Circular motion

Polar coordinates are a natural to describe circular motion. Consider a point P in moving in a circle as illustrated in Figure 4.23. If we choose O as the center of the circle, then r is simple the radius of the circle and is constant; hence

$$\dot{r} = 0 \qquad \text{and} \qquad \ddot{r} = 0\,.$$

Figure 4.23:

In describing circular motion, one sometimes introduces a new variable

$$\omega := \dot{\theta}\,.$$

Using the above expression for velocity in polar coordinates, we obtain that the velocity of P is given by

$$\bar{v} = v\hat{e}_\theta \qquad \text{where} \qquad v = r\omega\,.$$

Thus, the velocity of P is always tangential to the circle.

Figure 4.24: Velocity for circular motion

Noting that $\ddot{\theta} = \dot{\omega}$ and using the above above expression for acceleration in polar coordinates, we obtain that the acceleration of P is given by

$$\bar{a} = a_r\hat{e}_r + a_\theta\hat{e}_\theta \qquad \text{where} \qquad a_r = -r\omega^2 \text{ and } a_\theta = r\dot{\omega}$$

So the acceleration has both a radial and a tangential component. Since $v = rw$, we may express the acceleration as

$$\bar{a} = a_r\hat{e}_r + a_\theta\hat{e}_\theta \qquad \text{where} \qquad a_\theta = -\frac{v^2}{r} \text{ and } a_r = \dot{v}$$

Uniform circular motion. Suppose P is moving counter-clockwise in a circle at constant speed v. Then

$$\dot{v} = 0$$

and

$$\bar{a} = a_r\hat{e}_r \qquad \text{where} \qquad a_r = -\frac{v^2}{r}\,.$$

Thus, the acceleration of P is always towards the center of the circle. Sometimes this is called centripetal acceleration. The above expression also holds for clockwise motion.

Figure 4.25: Acceleration for uniform circular motion

4.5 General three-dimensional motion

4.5.1 Cartesian coordinates

$$\bar{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

4.5.2 Cylindrical coordinates

$$\rho, \theta, z$$

4.5.3 Spherical coordinates

$$r, \theta, \phi$$

Chapter 5

Kinematics of Reference Frames

5.1 Introduction

So far we have considered the kinematics of particles and points. Here we consider the kinematics of rigid bodies and reference frames.

A rigid body *is a body which has the property that the distance between every two particles of the body is constant with time.*

Although a rigid body is an idealized concept, it is a very useful concept in studying the motion of real bodies such as aircraft and spacecraft. To study the kinematics of a rigid body, we need only look at the kinematics of a reference frame in which the body is fixed; we call this a body fixed reference frame.

Figure 5.1: Body fixed frame

5.2 A classification of reference frame motions

Consider the motion of a reference frame g relative to another reference frame f. Suppose

$$g = (\hat{g}_1, \hat{g}_2, \hat{g}_3)$$

and G is the origin of g.

Figure 5.2: The motion of g in f

Translation. *The motion of g in f is a* translation *or g translates in f if the directions of $\hat{g}_1, \hat{g}_2, \hat{g}_3$ are constant in f.*

Figure 5.3: A translation

Thus, a translation can be completely characterized by the motion of the origin of f; hence the kinematics of translations can be completely described by the kinematics of points. A translation is said to be a rectilinear translation if the motion of the origin of g is rectilinear.

Rotation. *The motion of g in f is a* rotation *or g rotates in f if the origin of g is fixed in f. The motion of g in f is a* simple rotation *if there if a line L containing the origin of g which is fixed in both f and g.*

With regard to the above definition, we call L the axis of rotation and say that g rotates about L.

Figure 5.4: A rectilinear translation

Figure 5.5: A simple rotation

Fact. *Any rotation can be decomposed into at most three simple rotations.*

To illustrate the above fact consider the motion of g in f in the following picture where g is fixed in the bar. The motion of g in f is a rotation but it is not a simple rotation. Suppose one introduces a reference frame d which is fixed in the disc. Then the motion of g in f can be considered a composition of the motion of d in f followed by the motion of g in d. The latter two motions are simple rotations. Thus the motion of g in f is a composition of two simple rotations.

General Reference Frame Motions. *Any reference frame motion can be decomposed into a translation and a rotation.*

The above fact is illustrated by the motion of g in f in the following picture where g is fixed in the wheel. The motion of g in f is neither a translation nor a rotation. Suppose one introduces reference frame a which translates in f and whose origin is at the wheel center. Then the motion of g in f can be considered a composition of the motion of a in f and the motion of g in a, that is, it is a composition of a translation and a simple rotation.

Bar on cart

Figure 5.6: A composition of two simple rotations

Figure 5.7: A composition of a translation and a rotation

5.3 Motions with simple rotations

Recall that the motion of a reference frame g in a reference frame f can be decomposed into a translation and a rotation. In this section, we consider special motions, namely, motions whose rotational part is a simple rotation. As illustrated, we consider the motion of g in f; this motion being a composition of a translation and a simple rotation.

As a consequence of the above motion, there is a line containing G (the origin of g) which is fixed in g and has fixed orientation in f. We can call this the axis of rotation for the motion. In what follows we shall assume that $\hat{f}_3$ and $\hat{g}_3$ are chosen so that they are parallel to the axis of rotation. So, g moves in such a manner that $\hat{g}_3$ is always parallel to $\hat{f}_3$. The next concept is the most important concept in the kinematics of reference frames.

5.3.1 Angular Velocity

Let θ be the signed angle between $\hat{g}_1$ and $\hat{f}_1$ where θ is considered positive when $\hat{g}_1$ is counterclockwise of $\hat{f}_1$ (as viewed from the head of $\hat{f}_3$). Then the *angular velocity of g in f* is denoted by ${}^f\bar{\omega}^g$ and is defined by

$${}^f\bar{\omega}^g = \dot{\theta}\hat{f}_3 = \dot{\theta}\hat{g}_3 \,.$$

Note that ${}^f\bar{\omega}^g$ is a vector and is parallel to the axis of rotation of the motion. For practical purposes its direction can be determined by the *right-hand rule.* The quantity $\dot{\theta}$

Figure 5.8: A motion whose rotation is simple

Figure 5.9: Angular velocity

is called a *rate of rotation* and the *angular speed* of g in f is defined as $|\,{}^f\bar{\omega}^g| = |\dot{\theta}|$. If g translates in f, then ${}^f\bar{\omega}^g = \bar{0}$.

dim $[\,{}^f\bar{\omega}^g] = T^{-1}$
units: rad s^{-1}, rev min^{-1}
Fact. Suppose $\mathcal{B}$ is a rigid body and g and h are any two reference frames fixed in $\mathcal{B}$. Then, for any reference frame f, we have ${}^f\bar{\omega}^h = {}^f\bar{\omega}^g$ The above fact leads to the following definition for a rigid body $\mathcal{B}$.

Definition 1 The angular velocity of $\mathcal{B}$ in f,

$$ {}^f\bar{\omega}^{\mathcal{B}} := {}^f\bar{\omega}^g $$

where g is any reference frame fixed in $\mathcal{B}$.

Example 18 When viewed from above, a vinyl record $\mathcal{R}$ rotates clockwise at a the rate $\omega = 33\frac{1}{3}$rev/min. If reference frame f is fixed in the base of the record player, then

$$ {}^f\bar{\omega}^{\mathcal{R}} = -\omega \hat{f}_3 $$

Figure 5.10: Vinyl time

where

$$\omega = 33\frac{1}{3}\text{rev min}^{-1} = \frac{(33\frac{1}{3})(2\pi\text{rad})}{60\text{sec}} = 3.491\text{rad sec}^{-1}\,.$$

5.4 The Basic Kinematic Equation (BKE)

Suppose $\bar{Z}$ is any vector function of a scalar variable t and f and g are any two reference frames. Suppose that we are interested in ${}^f\dot{\bar{Z}}$, the derivative of $\bar{Z}$ in f, but for some reason

Figure 5.11: BKE

or other, it is more convenient to obtain ${}^g\dot{\bar{Z}}$, the derivative of $\bar{Z}$ in g. Can we relate ${}^g\dot{\bar{Z}}$ to ${}^f\dot{\bar{Z}}$? For the special motions considered in this section, the following theorem yields such a desired relationship.

Theorem 1 (Basic Kinematic Equation (BKE)) *If $\bar{Z}$ is any vector function of a scalar variable t, then*

$$\boxed{{}^f\dot{\bar{Z}} = {}^g\dot{\bar{Z}} + {}^f\bar{\omega}^g \times \bar{Z}}$$

Example 19 (Pendulum with moving support)

Example 20 (Bug on bar on cart)

5.4.1 Polar coordinates revisited

Recall that if we are studying the planar motion of a point P in a reference frame f, it is sometimes convenient to describe the motion of the point with polar coordinates, r and θ as illustrated. We previously derived expressions for $\bar{v}$, the velocity of P in f, and $\bar{a}$, the

Figure 5.12: Polar coordinates

acceleration of P in f, in terms of polar coordinates. Without the BKE, the derivation of these expressions was tedious. We now rederive these expressions using the BKE. This is one illustration of the usefulness of the BKE.

Figure 5.13: A useful reference frame for polar coordinates

We first introduce a new reference frame g which consists of the three mutually perpendicular unit vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_3$ as illustrated. Note that

$$^f\bar{\omega}^g = \dot{\theta}\hat{e}_3 \,.$$

The position of P relative to O is given by

$$\bar{r} = r\hat{e}_r \,.$$

To obtain the velocity of P in f, we use the BKE between reference frames f and g with $\bar{Z} = \bar{r}$ to yield

$$\bar{v} = \frac{^f d}{dt}(r\hat{e}_r) = \frac{^g d}{dt}(r\hat{e}_r) + {^f\bar{\omega}^g} \times (r\hat{e}_r) \,.$$

Since $\hat{e}_r$ is fixed in g,

$$\frac{^g d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r \,.$$

Also,

$$^f\bar{\omega}^g \times (r\hat{e}_r) = (\dot{\theta}\hat{e}_3) \times (r\hat{e}_r) = r\dot{\theta}\hat{e}_\theta$$

Hence,

$$\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \,.$$

To obtain the acceleration of P in f, we use the BKE between reference frames f and g with $\bar{Z} = \bar{v}$ to yield

$$\bar{a} = \frac{{}^f d\bar{v}}{dt} = \frac{{}^g d\bar{v}}{dt} + {}^f\bar{\omega}^g \times \bar{v}\,.$$

Since $\hat{e}_r$ and $\hat{e}_\theta$ are fixed in g,

$$\frac{{}^g d\bar{v}}{dt} = \frac{{}^g d}{dt}(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = \ddot{r}\hat{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

Also,

$${}^f\bar{\omega}^g \times \bar{v} = (\dot{\theta}\hat{e}_3) \times (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = \dot{r}\dot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r\,.$$

Hence,

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\,.$$

5.4.2 Proof of the BKE for motions with simple rotations

Consider two reference frames

$$f = (\hat{f}_1, \hat{f}_2, \hat{f}_3) \qquad \text{and} \qquad g = (\hat{g}_1, \hat{g}_2, \hat{g}_3)$$

and suppose that $\hat{f}_3$ and $\hat{g}_3$ are chosen so that $\hat{g}_3$ always has the same direction as $\hat{f}_3$.

Figure 5.14: Proof of BKE

Consider any vector Z. Since $\hat{g}_1, \hat{g}_2, \hat{g}_3$ constitute a basis, there is a unique triplet of scalars Z_1, Z_2, Z_3 such that

$$\bar{Z} = Z_1\hat{g}_1 + Z_2\hat{g}_2 + Z_3\hat{g}_3\,. \tag{5.1}$$

By definition,

$${}^g\dot{\bar{Z}} = \dot{Z}_1\hat{g}_1 + \dot{Z}_2\hat{g}_2 + \dot{Z}_3\hat{g}_3\,. \tag{5.2}$$

Utilizing (5.1) and (5.2), we obtain

$$\begin{aligned}
{}^f\dot{\bar{Z}} &= \frac{{}^f d}{dt}(Z_1\hat{g}_1 + Z_2\hat{g}_2 + Z_3\hat{g}_3) \\
&= \dot{Z}_1\hat{g}_1 + \dot{Z}_2\hat{g}_2 + \dot{Z}_3\hat{g}_3 + Z_1\frac{{}^f d\hat{g}_1}{dt} + Z_2\frac{{}^f d\hat{g}_2}{dt} + Z_3\frac{{}^f d\hat{g}_3}{dt} \\
&= {}^g\dot{\bar{Z}} + Z_1\frac{{}^f d\hat{g}_1}{dt} + Z_2\frac{{}^f d\hat{g}_2}{dt} + Z_3\frac{{}^f d\hat{g}_3}{dt}\,.
\end{aligned} \tag{5.3}$$

To compute $\frac{^f d\hat{g}_i}{dt}$ we need to express $\hat{g}_i$ in terms of the unit vectors of f.

$$\begin{aligned}
\hat{g}_1 &= C_\theta \hat{f}_1 + S_\theta \hat{f}_2 \\
\hat{g}_2 &= -S_\theta \hat{f}_1 + C_\theta \hat{f}_2 \\
\hat{g}_3 &= \hat{f}_3
\end{aligned}$$

Hence,

$$\frac{^f d\hat{g}_1}{dt} = -\dot{\theta} S_\theta \hat{f}_1 + \dot{\theta} C_\theta \hat{f}_2 = \dot{\theta}\hat{g}_2 \tag{5.4a}$$

$$\frac{^f d\hat{g}_2}{dt} = -\dot{\theta} C_\theta \hat{f}_1 - \dot{\theta} S_\theta \hat{f}_2 = -\dot{\theta}\hat{g}_1 \tag{5.4b}$$

$$\frac{^f d\hat{g}_3}{dt} = \bar{0} \tag{5.4c}$$

Looking at equations (5.4) and noting that

$$^f\bar{\omega}^g = \dot{\theta}\hat{f}_3 = \dot{\theta}\hat{g}_3 \ ,$$

we obtain

$$^f\bar{\omega}^g \times \hat{g}_1 = (\dot{\theta}\hat{g}_3) \times \hat{g}_1 = \dot{\theta}\hat{g}_2 = \frac{^f d\hat{g}_1}{dt} \tag{5.5a}$$

$$^f\bar{\omega}^g \times \hat{g}_2 = (\dot{\theta}\hat{g}_3) \times \hat{g}_2 = -\dot{\theta}\hat{g}_1 = \frac{^f d\hat{g}_2}{dt} \tag{5.5b}$$

$$^f\bar{\omega}^g \times \hat{g}_3 = (\dot{\theta}\hat{g}_3) \times \hat{g}_3 = \bar{0} = \frac{^f d\hat{g}_3}{dt} \tag{5.5c}$$

It follows from (5.5) that

$$\begin{aligned}
Z_1\frac{^f d\hat{g}_1}{dt} + Z_2\frac{^f d\hat{g}_2}{dt} + Z_3\frac{^f d\hat{g}_3}{dt} &= Z_1(\,^f\bar{\omega}^g \times \hat{g}_1) + Z_2(\,^f\bar{\omega}^g \times \hat{g}_2) + Z_3(\,^f\bar{\omega}^g \times \hat{g}_3) \\
&= \,^f\bar{\omega}^g \times (Z_1\hat{g}_1 + Z_2\hat{g}_2 + Z_3\hat{g}_3) \\
&= \,^f\bar{\omega}^g \times \bar{Z}\,.
\end{aligned} \tag{5.6}$$

Substitute (5.6) into (5.3) to obtain

$$^f\dot{\bar{Z}} = \,^g\dot{\bar{Z}} + \,^f\bar{\omega}^g \times \bar{Z}$$

■In a later section, we shall see that the BKE holds for arbitrary motions of g in f.

Exercises

Exercise 12 The two link planar manipulator is constrained to move in the plane defined by the vectors $\hat{e}_1$ and $\hat{e}_2$ of reference frame e. Point O is fixed in e.

Find *nice* expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$ in terms of θ_1, θ_2, their first and second time derivatives, l_1, l_2 and appropriate unit vectors.

Exercise 13 The small ball P moves in the straight slot which is fixed in the disk. Relative to reference frame e, point O is fixed and the disk rotates about an axis through O which is parallel to $\hat{e}_3$ and perpendicular to the plane of the disk.

Find *nice* expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$.

Exercise 14 The T-handle rotates at a constant rate Ω about a line fixed in reference frame e. Your favorite bug P is strolling along one leg of the handle.

Find *nice* expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$.

Exercise 15 Relative to reference frame e, the rigid frame $\mathcal{A}$ rotates at a constant rate rate Ω about a line passing through point O. Point P is moving along a line fixed in frame $\mathcal{A}$.

Find *nice* expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$.

Exercise 16 Relative to reference frame f, the disk of radius R is in simple rotation about an axis passing through point A; it oscillates according to

$$\phi(t) = \cos(t^2)\,.$$

The particle P is attached to point B on the disk by a taut string of constant length l.

Find *nice* expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$.

Exercise 17 The T-handle rotates at a constant rate of $\Omega = 100$ rpm about a line passing through point O and fixed in reference frame e. Your favorite bug B is strolling along one leg of the handle with a constant speed of $w = 5$ mph.

Given that $l = 5$ inches, *find* ${}^e\bar{a}^B$ when the bug reaches A.

Exercise 18 The pipe rotates at a constant rate of $\omega = 150$ rpm about a vertical line passing through point O and fixed in the grass. The small ball P is moving along the pipe with a constant speed of $\nu = 60$ ft/min.

Given that $\phi = 30\,\text{deg}$, *find* the acceleration of the ball relative to the grass when it reaches O.

Exercise 19 Relative to reference frame e, the large disk B rotates at a constant rate Ω about an axis which passes through point O and is parallel to $\hat{e}_3$. The small disk A rotates relative to B at a constant rate ω about an axis which passes through point Q and is parallel to $\hat{e}_3$. Your favorite bug P is strolling along a radial line fixed in A.

Find nice expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$.

Exercise 20 Relative to reference frame e, point P moves in the $\hat{e}_1 - \hat{e}_2$ plane with velocity $\bar{v}$. Assuming $\bar{v} \neq 0$, let $\hat{u}_v$ be the unit vector in the direction of $\bar{v}$; thus

$$\bar{v} = v\hat{u}_v$$

Also let $\hat{u}_\phi$ be the unit vector which is 90 degrees counterclockwise from $\hat{u}_v$.

Show that

$${}^e\bar{a}^P = \dot{v}\hat{u}_v + v\dot{\phi}\hat{u}_\phi$$

Chapter 6

General Reference Frame Motions

Here we consider the general motion of a reference frame g as seen by another reference f.

Figure 6.1: General reference frame motions

6.1 Angular velocity

The angular velocity of a reference g in another reference frame f (denoted ${}^f\bar{\omega}^g$) can be rigorously defined. Also the following property can be shown to hold.

If f, g, and h are any three reference frames, then

$$\boxed{{}^f\bar{\omega}^h = {}^f\bar{\omega}^g + {}^g\bar{\omega}^h}$$

In other words, angular velocities add up like position vectors.

Example 21 Propeller on pitching aircraft

Example 22 Pitching and rolling aircraft.

Here θ is the pitch angle of the aircraft and ϕ is the roll angle.

Using the above property, one can obtain the following general property

If $f^1, f^2, \ldots, f^n$ is any finite number of reference frames, then

$$\boxed{{}^{f^1}\bar{\omega}^{f^n} = {}^{f^1}\bar{\omega}^{f^2} + {}^{f^2}\bar{\omega}^{f^3} + \cdots + {}^{f^{(n-1)}}\bar{\omega}^{f^n}}$$

This relationship is very useful for practical computation of angular velocities. Recall that every rotation can be decomposed into at most three simple rotations. Since we know how to compute angular velocities for simple rotations, we can use the above relationship to compute angular velocities for general rotations.

Example 23

6.2 The basic kinematic equation (BKE)

The basic kinematic equation (BKE) holds for any motion of g in f. Specifically, if $\bar{Z}$ is any

Figure 6.2: The Basic Kinematic Equation (BKE)

vector function of time t, then

$$\boxed{\frac{{}^f d\bar{Z}}{dt} = \frac{{}^g d\bar{Z}}{dt} + {}^f\bar{\omega}^g \times \bar{Z}}$$

Example 24 Obtain expressions for ${}^e\bar{v}^P$ and ${}^e\bar{a}^P$.

Chapter 7

Angular Acceleration

Consider the motion of a reference frame g as seen by another reference frame f.

Figure 7.1: Angular acceleration

Angular acceleration is the time rate of change of angular velocity. The *angular acceleration of g in f* (denoted ${}^f\overline{\alpha}^{\,g}$) is defined as the time rate of change (in f) of the angular velocity of g in f, that is,

$$\boxed{{}^f\overline{\alpha}^{g} := \frac{{}^f d}{dt}\,{}^f\overline{\omega}^{\,g}}$$

If $\mathcal{B}$ is any rigid body and f is any reference frame, we denote the angular acceleration of $\mathcal{B}$ in f by ${}^f\overline{\alpha}^{\mathcal{B}}$ and define it to be equal to ${}^f\overline{\alpha}^{b}$ where b is any reference frame fixed in $\mathcal{B}$.

Figure 7.2: Angular acceleration of a rigid body

Example 25 (Two bars on a cart)

Figure 7.3: Two bars on a cart

• As illustrated in the previous example, for motions with simple rotations, the angular acceleration is always parallel to the angular velocity. In general, this is not true for general rotations; see the next example.

Some properties of angular acceleration. In general, one cannot add angular accelerations like angular velocities, that is,

$$^{f}\overline{\alpha}^{g} = {}^{f}\overline{\alpha}^{h} + {}^{h}\overline{\alpha}^{g} \qquad \text{FALSE}$$

This is illustrated in the next example.

Example 26 (Rotating pendulum)

Figure 7.4: Rotating pendulum

Example 27 (Propeller on pitching aircraft)

Figure 7.5: Propeller on pitching aircraft

- Another property:

$$\boxed{{}^f\overline{\alpha}^g = \frac{{}^g d}{dt}\,{}^f\overline{\omega}^g}$$

The above relationship says that one can obtain ${}^f\overline{\alpha}^g$ by differentiating ${}^f\overline{\omega}^g$ in the g frame instead of the f frame. In other words, the rate of change of ${}^f\overline{\omega}^g$ is the same for the frames f and g. To see this, use the definition of ${}^f\overline{\omega}^g$ and the BKE to obtain:

$$\begin{aligned}
{}^f\overline{\alpha}^g &= \frac{{}^f d}{dt}\,{}^f\overline{\omega}^g \\
&= \frac{{}^g d}{dt}\,{}^f\overline{\omega}^g + {}^f\overline{\omega}^g \times {}^f\overline{\omega}^g \\
&= \frac{{}^g d}{dt}\,{}^f\overline{\omega}^g
\end{aligned}$$

7.1 Exercises

Exercise 21 Relative to the earth, the cab of the crane is rotating at a constant rate $\Omega = 0.1$ rad/sec. Relative to the cab, the boom is being raised at a variable rate $\omega = 0.1t$ rad/sec. Find the magnitude of the angular acceleration of the boom relative to the earth when $t = 1$ sec.

Figure 7.6: A Crane

Find the magnitude of the angular acceleration of the boom relative to the earth when $t = 1$ sec.

Exercise 22 Relative to the earth, the propeller rotates at a constant rate Ω about the longitudunal axis of the aircraft. The aircraft is undergoing a maneuver in which its pitch angle θ is changing.

Figure 7.7: Aircraft

Obtain expressions for the angular velocity and the angular acceleration of the propeller relative to the earth.

Chapter 8

Kinematic Expansions

Figure 8.1: Kinematic expansions

8.1 The velocity expansion

Suppose we are interested in the velocity of a point P as seen by an observer in a reference f, but, it is much easier to obtain the velocity of P relative to another reference frame g. Can we relate ${}^f\bar{v}^P$ to ${}^g\bar{v}^P$? Yes, we can and that relationship is given by the velocity expansion (VE):

$$\boxed{{}^f\bar{v}^P = {}^f\bar{v}^G + {}^f\bar{\omega}^g \times \bar{r}^{GP} + {}^g\bar{v}^P} \tag{8.1}$$

where G is any point which is *fixed* in reference frame g.

Note that in applying the velocity expansion between f and g we must choose some convenient point G which is fixed in g. Quite often, G is the origin of g.

Examples and proof of VE.

8.2 The acceleration expansion

The acceleration expansion (AE) does for accelerations what the velocity expansion does for velocities. It is given by

$$\boxed{{}^f\bar{a}^P = {}^f\bar{a}^G + {}^f\bar{\omega}^g \times ({}^f\bar{\omega}^g \times \bar{r}^{GP}) + {}^f\bar{\alpha}^g \times \bar{r}^{GP} + 2\,{}^f\bar{\omega}^g \times {}^g\bar{v}^P + {}^g\bar{a}^P} \tag{8.2}$$

where G is any point which is *fixed* in reference frame g.

Examples and proof of AE.

8.3 Exercises

Exercise 23 The T-handle rotates at a constant rate of $\Omega = 100$ rpm about a line passing through point O and fixed in reference frame e. Your favorite bug B is sprinting along one leg of the handle with a constant speed of $w = 5$ mph.

Given that $l = 5$ inches, use the kinematic expansions to *find* ${}^e\bar{v}^B$ and ${}^e\bar{a}^B$ when the bug reaches A.

Exercise 24 The pipe rotates at a constant rate of $\omega = 150$ rpm about a vertical line passing through point O and fixed in the grass. The small ball P is moving along the pipe with a constant speed of $\nu = 60$ ft/min.

Given that $\phi = 30\,\text{deg}$, use the kinematic expansions to *find* the velocity and acceleration of the ball relative to the grass when it reaches O.

Exercise 25 The T-handle rotates at a constant rate Ω about a line fixed in reference frame e. Your favorite bug P is strolling along one leg of the handle.

Use the kinematic expansions to *find* nice expressions for ${}^{e}\bar{v}^{P}$ and ${}^{e}\bar{a}^{P}$.

Exercise 26 The disk rotates about a vertical axis through point O at a constant rate Ω. The small bug P moves relative to a slot fixed in the disk at a constant speed v. Considering a reference frame g fixed in the ground, find expressions for ${}^{g}\bar{v}^{P}$ and ${}^{g}\bar{a}^{P}$ at the instant P reaches point B.

Chapter 9

Particle Dynamics

9.1 Introduction

So far, we have considered motion without considering the causes of motion. We consider now what causes the motions of bodies. We initially look at the simplest types of bodies, namely, particles. Recall that a particle is a body that occupies a single point in space at each instant of time. Of course, this is a convenient mathematical idealization. However, it is very useful in modelling a physical body whose size is very small in comparison to other significant sizes in the situation under consideration. Consider the motion of the earth about the sun. In this situation, a good first approximation of the earth would be a particle. However, in studying the motion of an aircraft near the surface of the earth, a particle model of the earth is no longer useful. The point a particle occupies is called its position (or location). To study how motions are caused we need the concepts of mass and force.

The mass of a body is "a measure of its resistance to change in motion." The mass of a body is the same throughout the universe. Do not confuse the concepts of mass and weight. Mass is a positive scalar quantity. Its dimension is indicated by M and the SI and US units are kilogram (kg) and slug (slug), respectively.

Forces are the interactions between bodies. Every force is due to the interaction between two bodies. For a given body, the forces exerted on it by other bodies cause its motion. The effect of a force depends not only its magnitude and direction but also on where it is applied. So, we model forces with bound vectors. A bound vector is a vector which is associated with a specific point of application.

Figure 9.1: Bound vector

The point of application of a force acting on a particle is the position of the particle. The dimension of force is indicated by F. The SI and US units of force are the newton (N) and pound (lb), respectively, for example,

$$\bar{F} = (\hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3)\,\mathrm{lb}$$

or

$$\bar{F} = (2\hat{e}_1 + 3\hat{e}_3)\,\mathrm{N}\,.$$

9.2 Newton's second law

Consider a particle which is subject to a bunch of forces,

$$\bar{F}^1, \bar{F}^2, \ldots, \bar{F}^N \,,$$

as illustrated in Figure 9.2. Let $\Sigma\bar{F}$ be the resultant force acting *on* the particle, that is, it is the sum of *all* the forces acting on the particle. So,

$$\Sigma\bar{F} = \bar{F}^1 + \bar{F}^2 + \ldots + \bar{F}^N = \sum_{j=1}^{N} \bar{F}^j$$

Figure 9.2: Newton's second law

Sometimes Newton's second law is stated as follows: In an inertial reference frame, the acceleration of a particle is always proportional to the resultant force on the particle. This proportionality constant m is called the mass of the particle. So, we have

$$\boxed{\Sigma\bar{F} = m\bar{a}}$$

where $\bar{a}$ is the acceleration of the particle in an inertial reference frame. The problem with the above statement is that it introduces the undefined notion of an inertial reference frame. Another way to approach Newton's second law is first to define an inertial reference frame as any reference frame for which $\Sigma\bar{F} = m\bar{a}$ always holds and then simply state Newton's second law as:

There exists an inertial reference frame.

Practical inertial reference frames. For many everyday problems and problems in mechanical and civil engineering, a reference frame in which the earth is fixed can be considered inertial. However, in many aerospace situations, for example, a satellite orbiting the earth, one must use as inertial a reference frame with origin at the center of the earth and with respect to which the earth rotates at a rate of one revolution per day. If one is studying the motion of the earth relative to the sun, one must consider a reference fixed in the sun as inertial.

If f is an inertial reference frame and g is a reference frame which *translates with constant velocity* in f, then ${}^{g}\bar{a}^{P} = {}^{f}\bar{a}^{P}$; hence g is also inertial.

$\Sigma\bar{F} = m\bar{a}$ is not good for speeds close to the speed of light. In these situations, one must resort to relativistic mechanics.

From $\Sigma\bar{F} = m\bar{a}$, we must have

$$\begin{aligned} F &= MLT^{-2} \\ 1\mathrm{N} &= 1\,\mathrm{kg\,m\,s^{-2}} \\ 1\,\mathrm{slug} &= 1\,\mathrm{lb\,sec^{2}\,ft^{-1}} \end{aligned}$$

Suppose $\hat{b}_1, \hat{b}_1, \hat{b}_1$ are three basis vectors which are not necessarily orthogonal to each other and are not necessarily fixed in an inertial frame. Considering components relative to this basis, we have

$$\bar{a} = a_1\hat{b}_1 + a_2\hat{b}_2 + a_3\hat{b}_3$$

and

$$\Sigma\bar{F} = (\Sigma F_1)\,\hat{b}_1 + (\Sigma F_2)\,\hat{b}_2 + (\Sigma F_3)\,\hat{b}_3$$

where, for $i = 1, 2, 3$, the scalar ΣF_i is the sum of the components in the $\hat{b}_i$ direction of all the forces acting on the particle. Hence we obtain the following three scalar equations:

$$\boxed{\begin{aligned} \Sigma F_1 &= ma_1 \\ \Sigma F_2 &= ma_2 \\ \Sigma F_3 &= ma_3 \end{aligned}}$$

Basically, these equations state that *the resultant force in the i-th direction equals the mass times the acceleration in that direction.*

9.3 Static equilibrium

A particle P is in static equilibrium *if it is at rest in some inertial reference frame; that is, ${}^{f}\bar{v}^{P} = \bar{0}$ where f is inertial*

Result. If a particle is in static equilibrium, then the sum of all the forces acting on the particle is zero, that is,

$$\boxed{\Sigma\bar{F} = \bar{0}}$$

Proof. Since the particle is at rest in an inertial reference frame, its acceleration $\bar{a}$ in that frame is zero. The above result now follows from $\Sigma\bar{F} = m\bar{a}$ ■.

Suppose $\hat{b}_1, \hat{b}_1, \hat{b}_1$ are three basis vectors which are not necessarily orthogonal to each other and are not necessarily fixed in an inertial frame. Considering components relative to this basis, we have

$$\Sigma\bar{F} = (\Sigma F_1)\,\hat{b}_1 + (\Sigma F_2)\,\hat{b}_2 + (\Sigma F_3)\,\hat{b}_3$$

where, for $i = 1, 2, 3$, the scalar ΣF_i is the sum of the components in the $\hat{b}_i$ direction of all the forces acting on the particle. Hence we obtain the following three scalar equations:

$$\boxed{\begin{array}{rcl} \Sigma F_1 & = & 0 \\ \Sigma F_2 & = & 0 \\ \Sigma F_3 & = & 0 \end{array}}$$

Basically, these equations state that *the resultant force in the i-th direction equals zero.*

9.4 Newton's third law

Newtons Third Law has two parts:

(a) *If a body $\mathcal{A}$ exerts a force $\bar{F}$ on another body $\mathcal{B}$, then the second body $\mathcal{B}$ exerts a force $-\bar{F}$ on the first body $\mathcal{A}$ which is equal in magnitude but opposite in direction to $\bar{F}$.*

Sometimes this is loosely stated as "action and reaction are equal but opposite".

Figure 9.3: Newton's third law: first part

(b) *If $\mathcal{A}$ and $\mathcal{B}$ are particles then, the forces $\bar{F}$ and $-\bar{F}$ are along the line joining the two particles.*

Figure 9.4: Newton's third law: second part

9.5 Forces

As mentioned before, forces are interaction between bodies. A body is a collection of matter (solid, liquid, gas or a mixture of these states) which at each instant of time occupies some

region of space. In the following discussion of forces, we consider forces between arbitrary bodies; these bodies are not necessarily particles or rigid bodies.

We can divide forces into two types

(a) Contact forces are due to direct contact between bodies. One example is friction.

(b) Non-contact forces are exerted by bodies which are at a distance from each other and are not necessarily in contact. Examples include gravitational attraction and electromagnetic forces.

9.6 Gravitational attraction

9.6.1 Two particles

Universal law of gravitation. *If two particles of masses m_1 and m_2 are a distance r apart then, each attracts the other with a force of magnitude*

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

acting along the line joining the two particles where

$$G = 6.673 \times 10^{-11} \ m^3 \, kg^{-1} \, s^{-2}$$

is a universal constant.

Figure 9.5: Universal law of gravitation

The constant G is called the universal constant of gravitation.

9.6.2 A particle and a spherical body

Consider a particle of mass m and a spherical body of mass M and suppose that the particle is *outside* of the spherical body. Suppose also that the density at each point in the sphere only depends on the radial distance of that point from the center of the sphere. Then, applying the universal law of gravitation between the particle and every particle of the sphere and integrating over the sphere one can show that *the gravitational attraction of the sphere on the particle is equivalent to that of a particle of mass M located at the center of the sphere.* Thus, the resultant force exerted by the sphere on the particle has magnitude

$$F = G\frac{Mm}{r^2}$$

where r is the distance of the particle from the center of the sphere. This force is along the line joining the particle to the sphere center and is directed towards the sphere center.

Figure 9.6: Gravitational attraction of a sphere on a particle

9.6.3 A particle and the earth

Suppose we model the earth as a spherical body. Then, the gravitational attraction of the earth on a particle above earth is equivalent to that of a particle of mass $M_\oplus$ located at the center of earth where $M_\oplus$ is the mass of the earth and is given by

$$M_\oplus = 5.976 \times 10^{24} \text{ kg}.$$

Thus, the magnitude F of the resultant force exerted by the earth on a particle above the earth is given by

$$F = G\frac{M_\oplus m}{r^2}.$$

where r is the distance of the particle from the center of the earth. This force is along the line joining the particle to the center of the earth and is directed towards the center of the earth.

Figure 9.7: Gravitational attraction of the earth on a particle

Weight. If a particle is close to or on the surface of the earth then, $r \approx R_\oplus$ where $R_\oplus$ is the mean radius of the earth and is given by

$$R_\oplus = 6.371 \times 10^6 \text{ m}\,.$$

Thus, near the surface of the earth, the gravitational attraction of the earth on the particle is a force of magnitude

$$\boxed{W = mg}$$

where

$$g = \frac{GM_\oplus}{R_\oplus^2}\,.$$

This is called the weight of the particle. Substituting in the values for G, $R_\oplus$ and $M_\oplus$ we obtain

$$\boxed{g = 9.82 \text{ m}\,\text{s}^{-2} = 32.2 \text{ ft}\,\text{s}^{-2}}$$

The constant g is called the Earths surface gravitational constant.

Figure 9.8: Weight

As before the gravitational attraction of the earth is towards the center of the earth. This direction defines the local downward vertical direction which we usually indicate by the unit vector $\hat{g}$.

9.7 Contact forces

We idealize a contact force by idealizing the body exerting that force.

9.7.1 Strings

We idealize ropes, cables, etc., as strings.

A string *is a one-dimensional body. When it is* taut *and attached to another body, it exerts a force whose direction is tangential to the string and into the string at the point of attachment. If the string is not taut then, the force exerted by the string is zero.* The magnitude of the force exerted by a string is called the tension in the string.

Thus, a string pulls, but never pushes. Also, the direction of the force it exerts is completely determined by the string geometry. For a straight string, the force exerted by the string is parallel to the string and into the string.

Mathematically, we can represent a force $\bar{T}$ due to a string as

$$\bar{T} = T\hat{u}$$

where T is the tension in the string and the unit vector $\hat{u}$ is tangential to the string and into the string at the point of attachment.

Example 28 *Given:* The block of weight $W = 10$ lb is supported by two cables as shown.

Figure 9.9: Example 28

Find: the tension in each cable at the block.
Solution:

Example 29 *Given:* The ball of mass $m = 5$ kg is supported by two cables as shown.

Figure 9.10: Example 29

Find: the tension in each cable at the ball.
Solution:

9.7.2 Smooth surfaces

The force exerted by a smooth surface on a particle in contact with the surface is normal to the surface at the point of contact and directed towards the particle. (Thus the surface always pushes and never pulls.) Such a force is called a normal force.

Mathematically, we can represent this as

$$\bar{N} = N\hat{u} \qquad \text{where} \qquad N \geq 0$$

and the unit vector $\hat{u}$ is perpendicular to the surface at the particle and directed towards the particle.

9.7.3 Smooth curves

9.8 Free body diagrams and a systematic procedure

In order to reliably model the forces on a given body, a free body diagram is drawn. The free body diagram of a body is a figure containing the body under consideration (and no other bodies) and all the external forces acting on the body. All other bodies are replaced by the forces they exert on the body under consideration.

FBD should be drawn before applying $\Sigma\bar{F} = m\bar{a}$

The FBD should contain all available information on the forces. All forces should be labelled.

The following is a systematic procedure for applying $\Sigma\bar{F} = m\bar{a}$ to obtain scalar equations.

(a) *Obtain the inertial acceleration $\bar{a}$.* For static equilibrium problems, this is trivial, since $\bar{a} = 0$.

(b) *Draw a free body diagram (FBD) of the body.* This should contain all the external forces acting on the body and all available information regarding these forces.

(c) *Apply $\Sigma\bar{F} = m\bar{a}$.*

9.8.1 Springs

A spring is like a string except it can push as well as pull. When the spring is pulling, we say that it is in tension; when it is pushing, we say that it is in compression. We model a spring as a deformable one dimensional body of some length l. Usually springs are straight; but they can also be curvy. Graphical representations of springs are given in Figure 9.11.

Figure 9.11: Springs

Associated with any spring is its free length l_0; this is the length of the spring when it is not exerting any forces; thus is not subject to any forces. If we let $x = l - l_0$ then, x represents the amount by which the spring is extended ($x > 0$) or contracted ($x < 0$).

The force exerted by a spring attached to another body is tangential to the spring at the point of contact of the spring with the body. For a straight spring, the force is parallel to the spring. Thus the line of action of the force exerted by a spring is completely determined by the spring geometry. Thus we can represent a spring force by a single scalar S. In Figure 10.19, a positive value of S corresponds to the spring being in tension, while a negative value of S corresponds to the spring being in compression. The graph of S versus x for a spring is called the *characteristic curve* of the spring; see Figure 10.20. The simplest characteristic characteristic curve is linear, that is,

$$\boxed{S = kx}$$

for some positive scalar k. This sometimes refereed to as *Hookes Law*. The scalar is called the spring constant. Its units are force units/length units, for example N/mm or lbs/in .

Figure 9.12: Spring force

Figure 9.13: Spring characteristic curve

9.9 exercises

Exercise 27 The small ball of mass $m = 0.1$ kg rests on a hemisphere of radius $R = 0.25$m and is attached to point O via a string. *Find* the tension in the string.

Chapter 10

Forces

10.1 Introduction

Forces can be considered as the mechanical interactions between bodies. Sometimes it is convenient to classify forces into two types:

• *Contact forces* are forces which result from bodies being in contact with each other. Examples include *normal forces* and *friction*.

• *Forces which act at a distance.* The bodies do not have to be in contact with each other for these forces to be active. Examples include *gravitational forces* and *electromagnetic forces*.

In this chapter, we will see how to model various forces and utilize these models in application of Newton's second law. Before looking at these models, we present another basic law of mechanics, *Newton's third law.*

10.2 Newton's third law

Newtons third law has two parts: the first part applies to any two bodies and second part applies only to particles.

(a) *If a body exerts a force on another body, then the second body also exerts a force on the first body. The force exerted by the second body on the first is equal in magnitude but opposite in direction to the force exerted by the first body on the second.*

Figure 10.1: Newtons third law

(b) *When the two interacting bodies are particles, the two interacting forces are along the line joining the two particles.*

Figure 10.2: Newtons third law and particles

10.3 Gravitational forces

Every body in the universe is attracted to every other body via a gravitational force. For this force to be significant, one of the bodies must be "massive". Let us look first at the simplest situation, namely the gravitational interaction between two particles.

10.3.1 Two particles

Newton also had something to say here.

Universal law of gravitation. *If two particles of masses m_1 and m_2 are a distance r apart, then each attracts the other with a force of magnitude*

$$\boxed{F = \frac{Gm_1m_2}{r^2}}$$

acting along the line joining the particles where

$$G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$$

is a universal constant.

Figure 10.3: Universal law of gravitation

10.3.2 A particle and a spherical body

If the particle is outside the spherical body, then

$$F = \frac{GMm}{r^2}$$

Figure 10.4: A particle and a spherical body

10.3.3 A particle and YFHB

When the particle is near the surface of YFHB,

$$F \approx m\,g$$

where

$$g := \frac{GM}{R^2}$$

and R is the radius of YFHB.

10.4 Forces due to strings, ropes, etc.

Roughly speaking, by a string or a rope we mean a one-dimensional body which can bend without effort. The force $\bar{T}$ exerted by a *taut* string or rope on another body is given by

$$\boxed{\bar{T} = T\hat{u} \qquad T \geq 0}$$

where $\hat{u}$ is the unit vector which is tangential to the string at the body and is pointed into the string.

10.5 Application of $\Sigma\bar{F} = m\bar{a}$

10.5.1 Free body diagrams

In order to reliably model the forces on a given body, a free body diagram (FBD) is drawn. A free body diagram of a body is a picture containing the body and *all* the *external* forces acting on the body. All other bodies are replaced by the forces they exert on the given body.

$$\boxed{\textit{A FBD should be drawn before applying } \Sigma\bar{F} = m\bar{a}.}$$

10.5.2 A systematic procedure

(a) Obtain the inertial acceleration $\bar{a}$. In static problems, this is trivial because $\bar{a} = 0$.

(b) Model all the forces on the body. A free body diagram is needed at this step.

(c) Introduce a set of basis vectors and resolve the acceleration and all the forces into components relative to this basis.

(d) Apply $\Sigma\bar{F} = m\bar{a}$.

(e) Obtain scalar equations.

Sometimes step (c) is performed as part of step (e).

10.5.3 Examples

10.6 Forces due to surfaces and curves

10.6.1 Smooth surfaces

Consider a smooth small ball on a smooth flat table. What can you say about the direction

Figure 10.5: Ball on table

of the force exerted by the table on the ball?

Figure 10.6: Force exerted by table on ball

Consider YFI (your favorite insect) on YFBC (your favorite beverage can). Assuming the

Figure 10.7: YFI on YFBC

Figure 10.8: Force exerted by YFBC on YFI

can is cylindrical and there is no friction between YFI and YFBC, what can you say about the direction of the force exerted by YFBC on YFI?

Consider now the general situation of a particle in contact with a smooth surface. By a *surface*, we mean a 2-dimensional geometric object such as a plane or the "surface" of a cylinder. At every point on a surface (except at corners), there is a well defined line which passes through the point and is *normal* (perpendicular) to the surface at the point.

The force exerted by a smooth surface on a particle is normal to the surface at the position of the particle and is directed towards the particle. (Surfaces don't suck.) We can represent this by

$$\boxed{\bar{N} = N\hat{u} \qquad\qquad N \geq 0}$$

Figure 10.9: Particle on a surface

where the unit vector $\hat{u}$ is normal to the surface at the location of the particle and is directed towards the particle .

Figure 10.10: Normal force due to a surface

Thus the direction of the force exerted by a smooth surface is completely determined by the surface geometry and the location of the particle. This force is called a normal force.

Example 30

Example 31

10.6.2 Smooth curves

Consider a smooth bead constrained to move along a smooth straight wire.

Figure 10.11: Smooth bead on smooth wire

What can you say about the force exerted by the wire on the bead?

Figure 10.12: Force exerted by wire on bead

Consider a smooth small ball constrained to move inside a smooth circular tube .

Figure 10.13: Smooth ball in smooth tube

Figure 10.14: Force exerted by tube on ball

What can you say about the force exerted by the tube on the ball?

Consider now the general situation of a particle in contact with a curve. By a curve, we mean a 1-dimensional geometric object such as a straight line or a circle. At every point on a curve (except at corners), there is a well defined plane which passes through the point and is *normal* (perpendicular) to the curve at the point.

The force $\bar{N}$ exerted by a smooth curve on a particle is normal to the curve at the position of the particle, that is, it lies in the plane which is normal to the curve at the location of the

Figure 10.15: Particle in contact with a curve

particle. We can represent this by

$$\boxed{\bar{N} = N_2\hat{u}_2 + N_3\hat{u}_3}$$

where $\hat{u}_2, \hat{u}_3$ *are any two independent unit vectors which are normal to the curve at the location of the particle*

Figure 10.16: Normal force due to a curve

Note that in contrast to smooth surfaces, the direction of the force exerted by a smooth curve is unknown. This force is called a normal force.

Example 32

10.6.3 Rough surfaces and friction

Consider a small block on a rough flat table. What can we say about the force exerted by the table on the block?

Figure 10.17: Small block on a rough table

Consider now the general situation of a particle on a rough surface. The total force exerted by a rough surface on a particle is represented by

$$\boxed{\bar{N} + \bar{F}^f}$$

- The normal force $\bar{N}$ is normal to the surface at P and is towards P.
- The friction force $\bar{F}^f$ is tangential to the surface at P.

If we introduce a bunch of basis vectors $\hat{u}_1, \hat{u}_2, \hat{u}_3$ where $\hat{u}_1, \hat{u}_2$ are tangential to the surface at P and $\hat{u}_3$ is normal to the surface at P, then

$$\boxed{\begin{array}{rcll} \bar{F}^f & = & F_1^f \hat{u}_1 + F_2^f \hat{u}_2 & \\ & & & \\ \bar{N} & = & N \hat{u}_3 & \qquad\qquad N_3 \geq 0 \end{array}}$$

Plane surface

Cylindrical surface

General situation

Coulomb friction

A very common type of friction is Coulomb friction or dry friction. It usually occurs between two dry solid bodies in contact with each other. How do we walk? Why do motorcycles move?

Consider the general situation of a particle which is constrained to remain on a rough surface and suppose the friction force between the particle and the surface is due to Coulomb friction.

The Coulomb friction force depends on whether there is relative motion between the particle and the surface. To describe this, introduce a reference frame e in which the surface is fixed and let $\bar{v}$ be the velocity of the particle in e, that is, $\bar{v}$ is the velocoity of the particle relative to the surface, that is relative to e.

Static friction. Consider first the case in which there is no motion of the particle relative to the surface, that is $\bar{v} = \bar{0}$. Then the only additional statement that we can make about the friction force $\bar{F}^f$ exerted by the surface on the particle is that its magnitude F^f must satisfy the inequality,

$$\boxed{F^f \leq \mu N},$$

where $N = |\bar{N}|$ is the magnitude of the normal force exerted by the surface on the particle. The nonnegative constant $\mu = \mu_s$ is called a coefficient of static friction. It depends the surface properties of the objects in contact. Some examples are:

rubber on asphalt: $\mu_s = 0.85$

rubber on ice: $\mu_s = 0.1$

In static friction, the friction force is determined by the other forces on the particle and by the acceleration of the particle. In general,

$$F_f = \sqrt{(F_1^f)^2 + (F_2^f)^2}$$

In one of the components is zero then the expression for F^f is simpler, for example, if F_2^f is zero then,

$$F^f = |F_1^f|\,.$$

Sliding friction. Consider now the case in which there is motion of the particle relative to the surface, that is the particle is sliding on the surface and $\bar{v} \neq \bar{0}$. In this case the direction of the friction force $\bar{F}^f$ must be opposite to that of the velocity $\bar{v}$. Also the magnitude F^f of the friction force must satisfy the equality

$$\boxed{F^f = \mu N}$$

The non-negative constant $\mu = \mu_k$ is called a coefficient of kinetic friction. It depends on the surface properties of the objects in contact. Usually,

$$\mu_k < \mu_s$$

Why do you achieve maximum braking in a road vehicle by applying the brakes to the point where the wheels are just about to slip?

Example 33 How steep?

Example 34 What range?

Linear viscous friction

Sometimes, the friction between two lubricated bodies can be modelled as linear viscous friction. This type of friction force is opposite in direction to the velocity $\bar{v}$ and is proportional to $\bar{v}$. Mathematicaly, this can be expressed as

$$\bar{F}^f = -c\bar{v}$$

where the constant c is non-negative. This constant c is called a linear damping coefficient. Note that, unlike coulomb friction, this type of friction is zero when the velocity $\bar{v}$ is zero.

10.6.4 Rough curves

Similar to rough surfaces.

10.7 Springs

A common component in many machines and vehicles is a spring. A spring is like a string except it can push as well as pull. We model a spring as a massless deformable one dimensional body. of some length l. Usually springs are straight; but they can also be curvy. Graphical representations of springs are given in Figure 10.18.

Figure 10.18: Springs

Every spring has a rest length or unstretched length l_0. A spring exerts no force when it is at this length. When extended beyond its rest length, it pulls and we say that the spring is in tension. When compressed under its rest length, it pushes and we say the spring is in compression.

The force exerted by a spring attached to another body is tangential to the spring at the point of contact of the spring with the body. For a straight spring, the force is parallel to the spring. Thus the line of action of the force exerted by a spring is completely determined by the spring geometry. Thus we can represent a spring force by a single scalar S. Vectorially, we can represent a spring force by

$$\bar{F} = S\hat{u}$$

where $\hat{u}$ is a unit vector which is is tangential to the spring at its point of attachment and points into the spring; see Figure 10.19 A positive value of S corresponds to the spring being

Figure 10.19: Spring force

in tension, while a negative value of S corresponds to the spring being in compression; see Figure 10.20.

Figure 10.20: Sign convention for spring force

Figure 10.21: Spring characteristic curve

The scalar S depends only on the deformation of the spring from its unstretched state. Let x be the change in length of the spring from its rest length, that is, $x = l - l_0$ where l is the current length of the spring and l_0 is the *unstretched length* of the spring. Then, x represents the amount by which the spring is extended $(x > 0)$ or contracted $(x < 0)$. Also S is positive when x is positive and S is negative when x is negative.

The graph of S versus x for a spring is called the *characteristic curve* of the spring; see Figure 10.21. The simplest characteristic characteristic curve is linear, that is,

$$\boxed{S = kx}$$

for some positive scalar k. This is sometimes referred to as *Hookes Law.* The proportionality constant k is called the spring constant for the spring. It has dimension FL^{-1} and possible units are N/m or lbs/ft.

Figure 10.22: Linear spring

Nonlinear springs. Softening springs and hardening springs.

Example 35

10.8 Dashpots

By a dashpot we mean a one-dimensional massless deformable body with the property that the force it exerts depends only on its *rate* of extension or compression. Dashpots are useful for modeling many types of damping devices and the damping behavior of vehicle suspension components.

10.9 exercises

Exercise 28 The rough inclined plane is rotating about a vertical axis at a constant rate Ω. The small block of mass 0.1 kg rests on the inclined plane. The coefficient of static friction

Figure 10.23: Softening spring

Figure 10.24: Hardening spring

between the block and the plane is $\mu = 1/2$. If $\theta = 45°$ and $d = 100$ mm, determine the minimum and maximum values of Ω.

Chapter 11

Equations of Motion

In designing a spacecraft or aircraft, we like to know how it is going to behave before flying it. If aircraft and pilots were expendable like darts, one could probably design them totally by experimental trial and error, that is, dream up a design and fly the vehicle to see if it is cool or sucks. Then based on the outcome, make modifications and "fly" again. This is called the *Beavis and Butthead approach* to aerospace vehicle design. Many men–women and much expense would have been incurred in *trying* to land on the moon by this method. So, before flying an aerospace vehicle, we want to be able to predict its behavior as accurately as possible. This we do by developing a mathematical model of the vehicle. The most common model is a set of differential equations which describe the motion of the vehicle. These are called *equations of motion*. Of course, the concept of an equation of motion applies to any system described by the laws of mechanics. Actually, the idea of describing the behavior of a physical system using differential equation extends to all braches of engineering and science. It has even been used in economics. Let us begin with some simple systems.

11.1 Single degree of freedom systems

Example 1 (I'm falling) Consider particle P in vertical free fall near the surface of $YFHB$.

Neglecting any fluid resistance, application of $\Sigma \bar{F} = m\bar{a}$ in a vertical direction yields

$$\boxed{\ddot{x} = -g} \tag{11.1}$$

where g is the gravitational acceleration constant of YFHB. This equation is a *second order ordinary differential equation*. It has the property that, given any initial displacement x_0 and any initial rate of displacement v_0 at any initial time t_0, it has a unique solution for $x(t)$. Specifically, integrating (11.1) with respect to time over the interval $[0, t]$ yields

$$\dot{x}(t) - \dot{x}(0) = \int_0^t -g \, d\tau$$

hence,

$$\dot{x}(t) = v_0 - gt$$

Figure 11.1: I'm falling

where $v_0 = \dot{x}(0)$. Integrating again yields

$$x(t) - x(t_0) = v_0 t - \frac{g}{2} t^2$$

that is,

$$x(t) = x_0 + v_0 t - \frac{g}{2} t^2 \tag{11.2}$$

where $x_0 = x(0)$. From this last equation it should be clear that if x_0 and v_0 are specified then $x(t)$ is determined for all t, that is, the motion of P is completely determined by its initial position and velocity. Also *all* motions of P are given by this expression.

For the above reasons, equation (11.1) is called a (scalar) *equation of motion* (*EOM*) for P. Its solutions describe the manner in which x changes with time. Since the position of P is completely specified by x, this equation describes all possible motions of P.

Exercise 1 Show that if

$$x(t_0) = x_0 \qquad \dot{x}(t_0) = v_0$$

then $x(t)$ is given by (11.2) with t replaced by $t - t_0$ on the righthandside of (11.2), that is,

$$x(t) = x_0 + v_0 (t - t_0) - \frac{g}{2} (t - t_0)^2$$

Example 2 (The simple harmonic oscillator) The "small" block P of mass m moves without friction along a straight horizontal line fixed in YFHB. It is connected to point A by a linear spring of spring constant k and free length l_0. *Show* that the motion of P is described by

$$\boxed{m\ddot{x} + kx = 0}$$

Solution. We shall apply $\Sigma \bar{F} = m\bar{a}$ to P.

Choose reference frame e, fixed in YFHB, as inertial. Then,

$$\bar{a} = {}^e\bar{a}^P = \ddot{x}\hat{e}_1$$

Figure 11.2: Simple harmonic oscillator

Figure 11.3: Kinematics of the simple harmonic oscillator

Application of $\Sigma\bar{F} = m\bar{a}$ yields

$$-W\hat{e}_2 + N\hat{e}_2 - kx\,\hat{e}_1 = m\ddot{x}\,\hat{e}_1$$

The $\hat{e}_1$ components of the above equation yield:

$$\hat{e}_1 \;:\; -kx = m\ddot{x}$$

Hence,

$$m\ddot{x} + kx = 0 \tag{11.3}$$

■

The above equation is an *equation of motion* for P. It is a linear, second order, ordinary, differential equation. It has the property that, given any initial displacement x_0 and any initial rate of displacement v_0 at some initial time t_0, it has a unique solution for $x(t)$ at any other time t. In fact, with $t_0 = 0$, the solution is given by:

$$x(t) = x_0\cos(\omega t) + (v_0/\omega)\sin(\omega t) \tag{11.4}$$

where

$$x_0 \stackrel{\triangle}{=} x(0)\;; \qquad v_0 \stackrel{\triangle}{=} \dot{x}(0) \tag{11.5}$$

Figure 11.4: Free body diagram of P

Figure 11.5: Simple harmonic oscillator with an attitude

and

$$\omega \triangleq \sqrt{k/m} \tag{11.6}$$

Note that we can rewrite (11.3) as

$$\boxed{\ddot{x} + \omega^2 x = 0} \tag{11.7}$$

Exercise 2 Show that the above expression for $x(t)$ is the solution to (11.3) with initial conditions (11.5).

Exercise 3 (Simple harmonic oscillator with an attitude) Show that the motion of P is described by

$$\boxed{m\ddot{x} + kx + g\sin\theta = 0}$$

Example 3 (The simple pendulum) The simple pendulum consists of a particle P attached to a YFHB fixed point via a taut string of length l. It is constrained to move in a vertical plane. The position of P is completely specified by θ, the angle between the string and a vertical line. *Show* that the motion of P is governed by:

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

where g is the gravitational acceleration constant of YFHB.
SOLUTION. We shall apply $\Sigma\bar{F} = m\bar{a}$ to P.

Choose reference frame e, fixed in YFHB, as inertial. Then,

$$\bar{a} = {}^e\bar{a}^P = l\ddot{\theta}\hat{s}_1 + l\dot{\theta}^2\hat{s}_2$$

where reference frame s is fixed in the string.

Now applying $\Sigma\bar{F} = m\bar{a}$ yields

$$-mg\,\hat{e}_2 + T\,\hat{s}_2 = m(l\ddot{\theta}\hat{s}_1 + l\dot{\theta}^2\hat{s}_2)$$

where m is the mass of P. Since

$$\hat{e}_2 = \sin\theta\,\hat{s}_1 + \cos\theta\,\hat{s}_2$$

The simple pendulum

Figure 11.6: Kinematics of the simple pendulum

Figure 11.7: Free body diagram of pendulum bob

we obtain

$$\begin{array}{llll} \hat{s}_1: & -mg\sin\theta & = & ml\ddot{\theta} \\ \hat{s}_2: & -mg\cos\theta + T & = & ml\dot{\theta}^2 \end{array} \tag{11.8}$$

The first equation yields

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{11.9}$$

■

Note that application of $\Sigma\bar{F} = m\bar{a}$ in the above example yielded two equations. Suppose you were not given the EOM in the problem statement; why would you choose the first of equations (11.8) as the EOM? In general an EOM should only depend on the coordinate of interest (θ in this case), its first and second derivatives, and system parameters such as masses, spring constants, etc. Things like normal forces or string tensions should not appear in the final EOM. In the above example, one could use the second of the equations in (11.8) to solve for the string tension T as a function of P's motion. In general, if a particle is constrained to move along a curve (a circle in this example) application of $\Sigma\bar{F} = m\bar{a}$ in the the direction tangential to the curve yields the equation of motion.

The above EOM is a second order *nonlinear* differential equation. It has the property that for each set of initial conditions:

$$\theta(0) = \theta_0 \qquad \dot{\theta}(0) = \dot{\theta}_0$$

there is a unique solution $\theta(t)$ satisfying these conditions. However, since the equation is nonlinear, you cannot use the techniques (Laplace etc.) learned in *MA 262, MA 265, MA 266* to solve it. Although it is possible to obtain exact solutions to this equation, in general one cannot exactly solve nonlinear differential equations. To solve them one has to resolve to approximate *numerical techniques* .

A very special solution to (11.9) is the *equilibrium solution*

$$\theta(t) \equiv 0$$

which corresponds to initial conditions $\theta(0) = \dot{\theta}(0) = 0$. Suppose we are interested in the behaviour of the system near this equilibrium solution. For "small" θ,

$$\sin\theta \approx \theta$$

and the EOM (11.9) can be approximated by

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{11.10}$$

This looks familar, especially if we write it as

$$\ddot{\theta} + \omega^2\theta = 0$$

where

$$\omega = \sqrt{g/l}$$

(Recall (11.7)).

All of the systems considered so far were described by a single second order differential equation of the form

$$F(\ddot{x}, \dot{x}, x, t) = 0$$

This is because we only needed one coordinate to completely describe each of these systems. Such systems are called *single degree of freedom* systems. In a later section we look at *multi degree of freedom* systems.

11.2 Numerical simulation

11.2.1 First order representation

By appropriate definition of *state variables*

$$y_1, y_2, \ldots, y_n$$

one can rewrite any system of ordinary differential equations as a bunch of first order ordinary differential equations of the general form:

$$\begin{aligned}
\dot{y}_1 &= f_1(t, y_1, y_2, \ldots, y_n) \\
\dot{y}_2 &= f_2(t, y_1, y_2, \ldots, y_n) \\
&\vdots \\
\dot{y}_n &= f_n(t, y_1, y_2, \ldots, y_n)
\end{aligned}$$

Note that there is one equation for each state variable.

Example 36 *The simple harmonic oscillator.* Letting

$$y_1 = x \qquad y_2 := \dot{x}$$

this system has the first order representation:

$$\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\frac{k}{m} y_1
\end{aligned}$$

Example 37 *The simple pendulum.* With

$$y_1 := \theta \qquad y_2 := \dot{\theta}$$

this system has the first order representation:

$$\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\frac{g}{l} \sin y_1
\end{aligned}$$

11.2.2 Numerical simulation with MATLAB

```
>> help ode45

 ODE45  Solve differential equations, higher order method.
        ODE45 integrates a system of ordinary differential equations using
        4th and 5th order Runge-Kutta formulas.
        [T,Y] = ODE45('yprime', T0, Tfinal, Y0) integrates the system of
        ordinary differential equations described by the M-file YPRIME.M,
        over the interval T0 to Tfinal, with initial conditions Y0.
        [T, Y] = ODE45(F, T0, Tfinal, Y0, TOL, 1) uses tolerance TOL
        and displays status while the integration proceeds.

        INPUT:
        F     - String containing name of user-supplied problem description.
                Call: yprime = fun(t,y) where F = 'fun'.
                t      - Time (scalar).
                y      - Solution column-vector.
                yprime - Returned derivative column-vector; yprime(i) = dy(i)/dt.
        t0    - Initial value of t.
        tfinal- Final value of t.
        y0    - Initial value column-vector.
        tol   - The desired accuracy. (Default: tol = 1.e-6).
        trace - If nonzero, each step is printed. (Default: trace = 0).

        OUTPUT:
        T  - Returned integration time points (column-vector).
        Y  - Returned solution, one solution column-vector per tout-value.

        The result can be displayed by: plot(tout, yout).

        See also ODE23, ODEDEMO.
```

Example 38 Lets say we want to numerically simulate the simple pendulum over the time interval $0 \le t \le 20\,\text{sec}$ with parameters

$$l = 1 \qquad g = 1$$

and *initial conditions*

$$\theta(0) = \pi/2\,\text{rad} \qquad \dot{\theta}(0) = 0$$

We first write the equations in first order form; recall example 37. Next we create an M-file (lets call it `pendulum.m`) with the following lines.

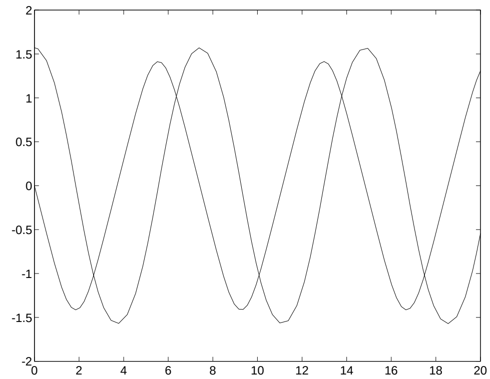
```
function ydot = pendulum(t,y)
ydot(1) = y(2)
ydot(2) = -sin(y(1))
```

We now simulate in MATLAB

```
>>[t,y]=ode45('pendulum',0,25,[pi/2; 0])
```

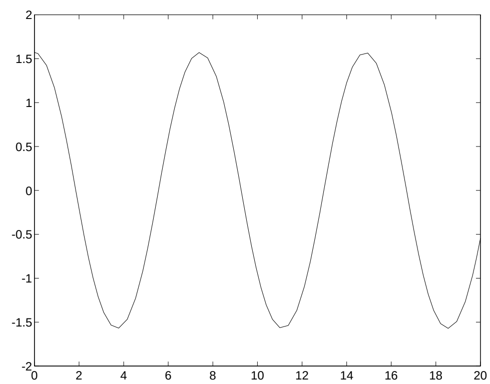
To get a plot:

```
>>plot(t,y)
```



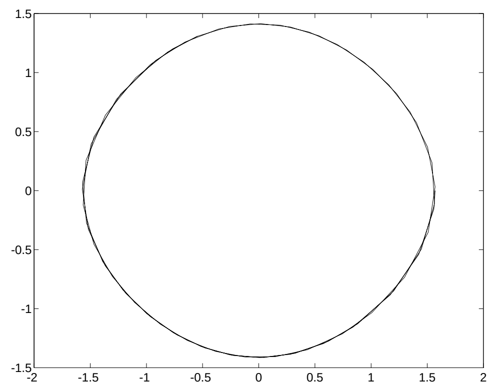
Now lets say we only want a plot of y_1 vs t.

```
>>plot(t,y(:,1))
```



Now suppose we want to plot y_1 vs. y_2. This is called a *phase plane* or *state plane* plot.

```
>>plot(y(:,1),y(:,2))
```


1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

11.3 Multi degree of freedom systems

Example 4 (The cannonball: ballistics in drag) Consider a cannonball P of mass m in flight in a vertical plane near the surface of the earth. Suppose we model the aerodynamic force on P as a force of magnitude $D(v)$ acting opposite to the velocity $\bar{v}$ of the cannonball relative to the earth and only dependent on the corresponding speed $v := |\bar{v}|$. *Show* that the motion of P is governed by

$$\boxed{\begin{array}{lcl} \dot{p} & = & v\cos\gamma \\ \dot{h} & = & v\sin\gamma \\ m\dot{v} & = & -mg\sin\gamma - D(v) \\ mv\dot{\gamma} & = & -mg\cos\gamma \end{array}} \tag{11.11}$$

where γ is the *flight path angle* and p and h are the *horizontal range* and *altitude* of P, respectively

Figure 11.8: Cannonball

SOLUTION. First note that

$$\begin{aligned} \bar{v} &= \frac{{}^e d}{dt}(\bar{r}^{OP}) \\ &= \frac{{}^e d}{dt}(p\,\hat{e}_1 + h\,\hat{e}_2) \\ &= \dot{p}\hat{e}_1 + \dot{h}\hat{e}_2 \end{aligned}$$

Also,

$$\bar{v} = v\cos\gamma\,\hat{e}_1 + v\sin\gamma\,\hat{e}_2$$

Comparing these two expressions for $\bar{v}$ yields

$$\begin{aligned} \dot{p} &= v\cos\gamma \\ \dot{h} &= v\sin\gamma \end{aligned}$$

Now we apply $\Sigma\bar{F} = m\bar{a}$ to P.

Reference frame u

Introduce reference frame

$$u = (\hat{u}_v,\ \hat{u}_\gamma,\ \hat{u}_3)$$

where $\hat{u}_v$ is the unit vector in the direction of $\bar{v}$, $\hat{u}_\gamma$ is the unit vector which is 90 degrees counterclockwise from $\hat{u}_v$, and $\hat{u}_3 = \hat{e}_3$. Then $\bar{v} = v\hat{u}_v$; ${}^e\bar{\omega}^u = \dot{\gamma}\hat{u}_3$; and application of the BKE between frames u and e yields

$$\begin{aligned}
\bar{a} \;:=\; & {}^e\bar{a}^P = \frac{{}^e d}{dt}(\bar{v}) \\
= \; & \frac{{}^u d}{dt}(\bar{v}) + {}^e\bar{\omega}^u \times \bar{v} \\
= \; & \dot{v}\,\hat{u}_v + v\dot{\gamma}\,\hat{u}_\gamma
\end{aligned}$$

Free body diagram of P

Application of $\Sigma\bar{F} = m\bar{a}$ yields:

$$\begin{array}{lrcl}
\hat{u}_v : & m\dot{v} & = & -mg\sin\gamma - D(v) \\
\hat{u}_\gamma : & mv\dot{\gamma} & = & -mg\cos\gamma
\end{array}$$

■

In the above example, the EOMs consisted of four first order differential equations. We could have obtained two second order differential equations in the coordinates p and h; however, they are not as nice as (11.11).

Exercise 4 For the above example, obtain two second order differential equations which describe the motion of P.

Exercise 5 (Sprung together) Consider a system consisting of two particles P_1 and P_2, connected together by a linear spring and constrained to move along a smooth horizontal line fixed in YFHB. Show that the motion of this system can be described by

$$\boxed{\begin{array}{l} m_1\ddot{q}_1 + k(q_1 - q_2) = 0 \\ m_2\ddot{q}_2 - k(q_1 - q_2) = 0 \end{array}}$$

Sprung together

11.4 Central force motion

A force is called a *central force* if its line of action always passes through an inertially fixed point. We call that point the *force center.* A particle is said to undergo *central force motion* if the only force acting on it is a central force.

Central force motion

Example 5 *The table oscillator*

Consider particle P which is constrained to move on the surface of a smooth horizontal table and is attached to inertially fixed point O by a linear spring of spring constant k and free length l_0.

The table oscillator

Consideration of $\Sigma\bar{F} = m\bar{a}$ in a vertical direction shows that the normal force cancels out the weight force.

Free body diagram of P

Hence, the original free body diagram of P is equivalent to the next free body diagram.

Thus P undergoes a central force motion where the central force is the spring force.

Equivalent free body diagram of P

Example 6 *Some orbit mechanics*
Consider a body P of mass m in orbit about YFHB $\mathcal{B}$ of mass M.

Some orbit mechanics

Modelling $\mathcal{B}$ as a sphere whose mass density depends only on the distance from its center, the gravitational attaction of $\mathcal{B}$ on P is a force directed towards the center of $\mathcal{B}$. If we consider the situation in which $\mathcal{B}$ is relatively massive in comparison to P, that is,

$$M \gg m$$

then we may regard the center of $\mathcal{B}$ as inertially fixed. Hence P undergoes a central force motion. Examples of this include:

$\mathcal{B}$	P
earth	you
earth	satelite
moon	you
sun	earth

11.4.1 Equations of motion

We shall see later that every central force motion is a planar motion and the plane of motion must contain the force center. The plane is determined by an initial position and initial velocity of the particle. We use polar coordinates (r, θ) to describe central force motion.

Kinematics of central force motion

The inertial acceleration of P is given by

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$$

Free body diagram of P

Applying $\Sigma\bar{F} = m\bar{a}$ yields

$$-F\hat{e}_r = m\bar{a}$$

Looking at the $\hat{e}_r$ and $\hat{e}_\theta$ components of $\Sigma\bar{F} = m\bar{a}$ yields the following two EOMs:

$$\boxed{\begin{array}{ccccccc} \ddot{r} & - & r\dot{\theta}^2 & + & F/m & = 0 \\ r\ddot{\theta} & + & 2\dot{r}\dot{\theta} & & & = 0 \end{array}}$$

11.4.2 Some orbit mechanics

Consider the motion of a small body P of mass m about a much larger spherical body $\mathcal{B}$ of mass M. We can regard the center of the spherical body as inertially fixed; hence the motion of the smaller body is a central force motion with

$$F = \frac{GMm}{r^2}$$

Some orbit mechanics

Recalling the above EOMs for general central force motion, the motion of P is described by

$$\boxed{\begin{array}{ccccccc} \ddot{r} & - & r\dot{\theta}^2 & + & \mu/r^2 & = 0 \\ r\ddot{\theta} & + & 2\dot{r}\dot{\theta} & & & = 0 \end{array}}$$

where $\mu := GM$.

All solutions of the above two differential equations are *conic sections*; that is they satisfy an equation of the form:

$$r = \frac{a}{1 + b\cos(\theta - c)}$$

The constants a, b, and c depend on the motion.

$$\begin{array}{rl} b = 0 & \text{circle} \\ 0 < b < 1 & \text{ellipse} \\ b = 1 & \text{parabola} \\ b > 1 & \text{hyperbola} \end{array}$$

For the moment, we will only look at circular orbits. AAE 340 contains a closer look at all orbits. AAE 532 *(Orbit mechanics)* is a whole course devoted to orbit mechanics.

Circular orbits

Let's look for solutions corresponding to circular orbits, that is,

$$r(t) \equiv R$$

where R is constant; hence $\ddot{r} = \dot{r} = 0$. The above EOMs reduce to

$$\begin{array}{cccc} -R\dot{\theta}^2 & + & \mu/R^2 & = 0 \\ R\ddot{\theta} & & & = 0 \end{array}$$

The second equation implies that $\dot{\theta}$ is constant, that is,

$$\dot{\theta}(t) \equiv \omega$$

where ω is constant. The first equation now implies that

$$R^3\omega^2 = \mu$$

hence

$$\omega = \sqrt{\mu/R^3} \qquad \text{or} \qquad R = (\mu/\omega^2)^{1/3} \tag{11.12}$$

Geostationary orbits

Suppose one wants to position a satellite so that it always remains above a fixed point on the earth. To study this motion, we need to take an inertial reference frame in which the earth rotates about its north-south axis at the rate:

$$\begin{aligned} \omega_e &= 1\text{rev}/24\,\text{hour} \\ &= \frac{(2\pi\text{rad})}{(24)(60)(60)\,\text{sec}} \\ &= 7.272 \times 10^{-5}\,\text{rad/sec} \end{aligned}$$

Inertial reference frame

Since the satellite must move in a plane which contains the center of the earth, it must be located above the equator.

Geostationary orbit

Using (11.12), the satellite must be located at the following distance from the center of the earth:

$$\begin{aligned} R &= \left[GM_{earth}/\omega_e^2\right]^{1/3} \\ &= \left[\frac{(6.673\times10^{-11})(5.976\times10^{24})}{(7.272\times10^{-5})^2}\right]^{1/3} \\ &= 42,247\,\mathrm{km} \end{aligned}$$

Hence, the satellite must be located at a height

$$\begin{aligned} h &= R - R_{earth} \\ &= 42,247 - 12,755/2 \\ &= 35,870\,\mathrm{km} \end{aligned}$$

Chapter 12

Statics of Bodies

Prior to this, we have considered the statics of particles. In this chapter, we consider the statics of general bodies. First, we need a new concept which is basic in the study of the statics and dynamics of bodies, namely, the moment of a force.

12.1 The moment of a force

The moment of a force about a point is its turning effect about that point. As an example, think of a person pushing or pulling on one end of a joystick and consider the turning effect of this force about the pivot point at the other end of the joystick. The formal definition of a moment is as follows.

The moment of a force *$\bar{F}$ about a point Q is defined by*

$$\boxed{\bar{M}^Q = \bar{r} \times \bar{F}}$$

where $\bar{r} = \overline{QP}$ (the vector from Q to P) and P is the point of application of $\bar{F}$.

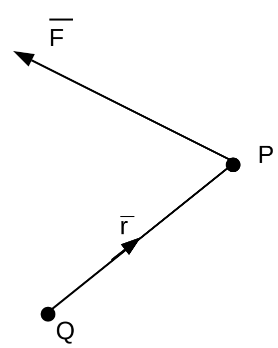


Figure 12.1: Moment of a force

• Recalling the definition of the cross product, it follows that

$$\boxed{\bar{M}^Q = M^Q\,\hat{n} \quad \text{where} \quad M^Q = rF\sin\theta}$$

Here r is the distance from Q to the point of application of the force, F is the magnitude of the force, θ is the angle between $\bar{r}$ and $\bar{F}$, and $\hat{n}$ is the unit vector which is normal to both $\bar{r}$ and $\bar{F}$ and whose sense is given by the right-hand rule; see Figure 12.2. Note that M^Q is the magnitude of $\bar{M}^Q$.

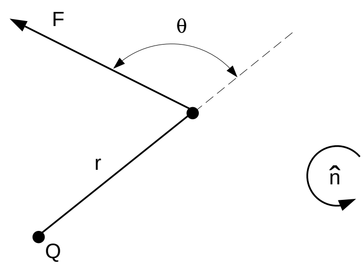


Figure 12.2: Moment of a force again

- Units: newton-meter (N·m) or foot-pound (ft·lb).

Example 39

The next fact tells us that in evaluating the moment of a force, we can choose the position vector to terminate at any point on the line of action of the force.

Fact 2 *If P is any point on the line of action of $\bar{F}$, then*

$$\bar{M}^Q = \bar{r} \times \bar{F}$$

where $\bar{r} = \overline{QP}$ (the vector from Q to P).

Figure 12.3: Any point on the line of action will do.

Proof. By definition,

$$\bar{M}^Q = \overline{QP^*} \times \bar{F}$$

where P^* is the point of application of $\bar{F}$. Since

$$\overline{QP^*} = \overline{QP} + \overline{PP^*}$$

we have

$$\bar{M}^Q = \overline{QP} \times \bar{F} \ + \ \overline{PP^*} \times \bar{F}$$

Since the points P and P^* are on the line of action of $\bar{F}$, the vector $\overline{PP^*}$ is along the line of action of $\bar{F}$; hence

$$\overline{PP^*} \times \bar{F} = \bar{0}$$

and

$$\bar{M}^Q = \overline{QP} \times \bar{F}$$

■

Example 40

The following fact is useful for determining moments by inspection, especially in planar problems.

Fact 3 *If d is the distance from a point Q to the line of action of a force of magnitude F, then the magnitude M^Q of the moment of that force about Q is given by*

$$M^Q = dF\,.$$

Figure 12.4: $M^Q = dF$

PROOF. Let P be the point on the line of action of $\bar{F}$ which is closest to Q. Then the vector $\overline{QP}$ is perpendicular to $\bar{F}$ and the magnitude of this vector is d. Using the first definition of the cross product, we have

$$\begin{aligned} M^Q = |\bar{M}^Q| &= |\overline{QP} \times \bar{F}| \\ &= |\overline{QP}||\bar{F}| \sin(90^\circ) \\ &= dF \end{aligned}$$

■

Note that the direction of the moment is determined by the right-hand rule.

Example 41

The next result is an immediate consequence of the previous fact.

Fact 4 *Suppose $\bar{F}$ is a nonzero force. Then its moment about a point Q is zero if and only if Q is on the line of action of $\bar{F}$.*

12.2 Bodies

A **physical body** is any material object. It can be solid, liquid, gas or a combination of these. A piece of a body is just another body. A collection of bodies can also be regarded as a single body. Mathematically, a body is something with two properties:

i) At each instant of time, it occupies a region of space.

ii) It has a *mass distribution*; to each piece of the body, we can associate a real number, the mass of that piece.

• A **particle** is the simplest type of body; at each instant of time, it occupies a single point.

• An arbitrary body can be regarded as a collection of particles.

• A **rigid body** can be defined as a body with the property that the distance between any two particles of the body always remains the same.

12.2.1 Internal forces and external forces

All the forces acting on a particle act at a single point, namely, the position of the particle.

Figure 12.5: Forces on a particle

The forces acting on a general body do necessarily not act at a single point. They can act at any point in the region of space the body occupies.

Consider a general body $\mathcal{B}$. We can classify the forces associated with $\mathcal{B}$ into two types:

• Forces **internal** to the body $\mathcal{B}$. An internal force is a force exerted by one piece of $\mathcal{B}$ on another piece of $\mathcal{B}$.

• Forces **external** to the the body $\mathcal{B}$. These are the forces exerted on $\mathcal{B}$ by other bodies.

Figure 12.6: Forces on a general body

12.2.2 Internal forces

By considering a body as a collection of particles and applying Newton's third law, one can obtain the following result.

Theorem 2 (Internal forces) *The internal forces of any body satisfy*

$$\boxed{\begin{array}{rcl} \sum^{int} \bar{F} & = & \bar{0} \\ \\ \sum^{int} \bar{M}^{Q} & = & \bar{0} \end{array}}$$

for every point Q where

$\sum^{int} \bar{F}$ *is the sum of all the* internal *forces in the body, and*

$\sum^{int} \bar{M}^{Q}$ *is the sum of the moments about Q of all the* internal *forces in the body*

PROOF. Consider a general body $\mathcal{B}$ which we will regard as a collection of N particles $P_1, P_2, \ldots, P_N$.

Let $\bar{F}^{jk}$ be the resultant internal force exerted on particle P_j by particle P_k.

Thus the internal force system of $\mathcal{B}$ consists of the forces

$$\bar{F}^{jk}, \qquad j \neq k, \quad j,k = 1,2,\ldots,N$$

First note that the resultant of all the internal forces is given by

$$\begin{aligned}
\sum^{int} \bar{F} &= \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \bar{F}^{jk} \\
&= \begin{array}{ccccccccccc}
 & \bar{0} & + & \bar{F}^{12} & + & \bar{F}^{13} & + & \ldots & + & \bar{F}^{1N} \\
+ & \bar{F}^{21} & + & \bar{0} & + & \bar{F}^{23} & + & \ldots & + & \bar{F}^{2N} \\
+ & \bar{F}^{31} & + & \bar{F}^{32} & + & \bar{0} & + & \ldots & + & \bar{F}^{3N} \\
 & \vdots & & \vdots & & \vdots & & & & \vdots \\
+ & \bar{F}^{N1} & + & \bar{F}^{N2} & + & \bar{F}^{N3} & + & \ldots & + & \bar{0}
\end{array} \\
&= \begin{array}{ccccccc}
(\bar{F}^{12} + \bar{F}^{21}) & + & (\bar{F}^{13} + \bar{F}^{31}) & + & \ldots & + & (\bar{F}^{1N} + \bar{F}^{N1}) \\
 & + & (\bar{F}^{23} + \bar{F}^{32}) & + & \ldots & + & (\bar{F}^{2N} + \bar{F}^{N2}) \\
 & & & & & + & (\bar{F}^{3N} + \bar{F}^{N3}) \\
 & & & & & & \vdots \\
 & & & & & + & \bar{0}
\end{array} \\
&= \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} (\bar{F}^{jk} + \bar{F}^{kj})
\end{aligned}$$

By the first part of *Newton's Third Law*, $\bar{F}^{kj} = -\bar{F}^{jk}$;

hence

$$\bar{F}^{jk} + \bar{F}^{kj} = \bar{0}$$

Thus,

$$\boxed{\sum^{int} \bar{F} = \bar{0}}$$

Consider now any corresponding pair of internal forces $\bar{F}^{jk}$, $\bar{F}^{kj}$. By the second part of *Newtons Third Law* the line of action of these two forces must be the line joining P_j and P_k; hence the vector $\overline{P_k P_j}$ is parallel to $\bar{F}^{jk}$ and $\bar{F}^{kj}$.

So,

$$\overline{P_k P_j} \times \bar{F}^{jk} = \bar{0}$$

Evaluating the sum of the moments of these two forces about any point Q and using $\bar{F}^{kj} = -\bar{F}^{jk}$ we get

$$\begin{aligned} \bar{r}^j \times \bar{F}^{jk} \; + \; \bar{r}^k \times \bar{F}^{kj} &= (\bar{r}^j - \bar{r}^k) \times \bar{F}^{jk} \\ &= \overline{P_k P_j} \times \bar{F}^{jk} \\ &= \bar{0} \end{aligned}$$

Hence, the moments due to the internal forces cancel out in pairs. If $\sum^{int} \bar{M}^Q$ is the sum of the moments of all the internal forces about Q, then using the same computations we used for $\sum^{int} \bar{F}$ we must have

$$\boxed{\sum^{int} \bar{M}^Q = \bar{0}}$$

■

• Note that the above result holds regardless of the motion of the body; the body does not have to be in static equilibrium (see next section).

12.3 Static equilibrium

Recall that a particle is in static equilibrium if it is at rest in some inertial reference frame. If we regard an arbitrary body as a collection of particles, we have the following definition.

Defn. *A body is in* static equilibrium *if every particle of the body is at rest in the same inertial reference frame.*

Our next result is the most important result in the statics of bodies. It can be obtained by applying Newton's second law to each particle of a body, summing over all the particles in the body, and using the fact that the internal forces and moments sum to zero.

Theorem 3 *If a body is in static equilibrium, then for any point Q*

$$\boxed{\begin{aligned} \sum \bar{F} &= \bar{0} \\ \sum \bar{M}^Q &= \bar{0} \end{aligned}}$$

where

$\sum \bar{F}$ *is the sum or resultant of all the* external *forces acting on the body*

$\sum \bar{M}^Q$ *is the sum of the moments or the moment resultant about Q of all the* external *forces acting on the body*

12.3.1 Free body diagrams

$$\boxed{\text{Body} \quad + \quad \text{External forces}}$$

Examples of FBDs

12.4 Examples in static equilibrium

12.4.1 Scalar equations of equilibrium

In general, the two vector equations

$$\begin{aligned}\Sigma\bar{F} &= \bar{0}\\ \Sigma\bar{M}^Q &= \bar{0}\end{aligned}$$

yield six scalar equations. However, in some cases, the six equations are not independent or some are trivial, for example, $0 = 0$. The following are force systems which do not give rise to six independent scalar equations of equilibrium.

Force system	Max. no. of independent scalar equations
collinear	1
coplanar	3
parallel	3
parallel to a common plane	5

12.4.2 Planar examples

A planar problem is one in which all the bodies and forces of interest lie in a single plane. Suppose $\hat{e}_1, \hat{e}_2, \hat{e}_3$ is an orthogonal triad of unit vectors with $\hat{e}_1, \hat{e}_2$ lying in the plane of interest and with $\hat{e}_3$ perpendicular to the plane. Then all forces and position vectors can be expressed in terms of $\hat{e}_1$ and $\hat{e}_2$; hence all moments are parallel to $\hat{e}_3$. So, the conditions of static equilibrium give rise to at most three nontrivial scalar equations:

$$\begin{aligned}\hat{e}_1: &\quad \sum F_1 = 0\\ \hat{e}_2: &\quad \sum F_2 = 0\\ \hat{e}_3: &\quad \sum M^Q = 0\end{aligned}$$

12.4.3 General examples

12.5 Force systems

A force system is just a bunch of forces, $\bar{F}^1$, $\bar{F}^1$, . . . , $\bar{F}^N$.

Figure 12.7: A force system

Defn. The resultant of a force system is the sum of all its forces and is given by:

$$\sum \bar{F} \quad := \quad \sum_{j=1}^{N} \bar{F}^j \quad = \quad \bar{F}^1 + \bar{F}^2 + \ldots + \bar{F}^N$$

Defn. The moment resultant of a force system about a point Q is the sum of the moments of all its forces about Q and is given by:

$$\sum \bar{M}^Q \quad := \quad \sum_{j=1}^{N} \bar{r}^j \times \bar{F}^j \quad = \quad \bar{r}^1 \times \bar{F}^1 \,+\, \bar{r}^2 \times \bar{F}^2 \,+\, \ldots \,+\, \bar{r}^N \times \bar{F}^N$$

where $\bar{r}^j$ is a vector from Q to a point on the line of action of $\bar{F}^j$.

Figure 12.8: Moment resultant

Fact 5 *For any two points Q and Q',*

$$\boxed{\sum \bar{M}^Q = \sum \bar{M}^{Q'} + \bar{r} \times \Sigma \bar{F}}$$

where $\bar{r} = \bar{r}^{QQ'}$ (the vector from Q to Q').

Proof. By definition we have

$$\sum \bar{M}^Q = \sum_{i=1}^{N} \bar{r}^j \times \bar{F}^j \qquad \text{and} \qquad \sum \bar{M}^Q = \sum_{i=1}^{N} \bar{\rho}^j \times \bar{F}^j$$

Figure 12.9: Q and Q'

where $\bar{r}^j$ is the vector from Q to the point of application of $\bar{F}^j$ and $\bar{\rho}^j$ is the vector from Q' to the point of application of $\bar{F}^j$. However $\bar{r}^j = \bar{\rho}^j + \bar{r}$ where $\bar{r} = \bar{r}^{QQ'}$; hence

$$\begin{aligned}
\sum \bar{M}^Q &= \sum_{i=1}^{N} (\bar{\rho}^j + \bar{r}) \times \bar{F}^j \\
&= \sum_{i=1}^{N} \bar{\rho}^j \times \bar{F}^j \quad + \quad \bar{r} \times \sum_{i=1}^{N} \bar{F}^j \\
&= \sum \bar{M}^{Q'} + \bar{r} \times \Sigma \bar{F} \qquad \blacksquare
\end{aligned}$$

Example 42

12.5.1 Couples and torques

Defn. A couple is a pair of forces which have equal magnitude but opposite direction.

Figure 12.10: A couple

So, if $(\bar{F}^1, \bar{F}^2)$ is a couple, then,

$$\bar{F}^2 = -\bar{F}^1 .$$

• A couple has zero resultant.

• The moment resultant of a couple about every point is the same.

DEFN. The torque $\bar{T}$ of a couple is its moment resultant about any point.

Quite often, we are only interested in the torque of a couple and we are not necessarily interested in the two forces that make up the couple. So, we often represent a couple by its torque $\bar{T}$. In graphic representations of torque vectors, we use a double arrowhead instead of the usual single arrowhead.

Example 43

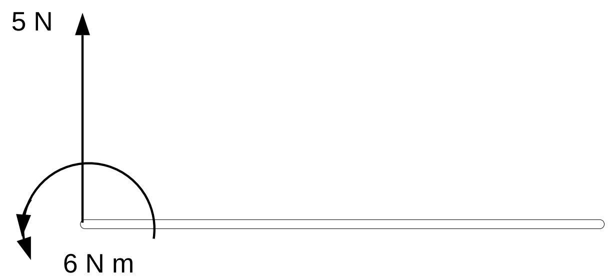
12.6 Equivalent force systems

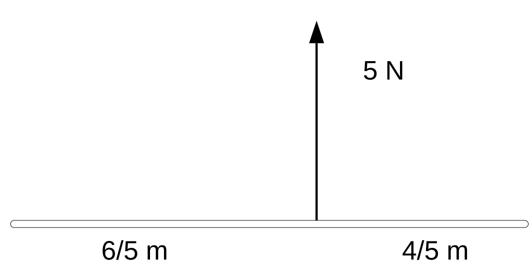
Defn. *Two force systems are* equivalent *if they have the same resultant and the same moment resultant about some point.*

So, the system of internal forces in any body is equivalent to a zero force. Also, if a body is in static equilibrium, its system of external forces is equivalent to a zero force.

Example 44 Consider the following force system.

It is equivalent to any of the following force systems.





• *If two force systems are equivalent, then they have the same moment about every point.* This follows from the relationship, $\sum \bar{M}^Q = \sum \bar{M}^{Q'} + \bar{r} \times \Sigma \bar{F}$.

You will see later that if two force systems are equivalent, then they have exactly the same effect when applied to a given rigid body.

12.7 Simple equivalent force systems

Sometimes it is very useful to replace a complicated force system by a simpler equivalent force system.

12.7.1 A force and a couple

Every force system (regardless of complexity) is equivalent to a force and a couple. To see this, choose *any* point Q and let

$$\bar{F} = \Sigma \bar{F} \qquad \text{and} \qquad \bar{T} = \Sigma \bar{M}^Q$$

It can readily seen that the new force system consisting of a force $\bar{F}$ placed at Q and a couple of torque $\bar{T}$ is equivalent to the original force system.

Note that Q can be any point.

12.7.2 Force systems which are equivalent to a couple

Suppose

$$\Sigma \bar{F} = \bar{0}$$

Then the original force system is equivalent to a couple of torque $\bar{T} = \Sigma \bar{M}^Q$. Since $\Sigma \bar{F} = \bar{0}$, this torque is independent of the point Q.

12.7.3 Force systems which are equivalent to single force

Suppose there is a point Q such that

$$\Sigma \bar{M}^Q = \bar{0}$$

Then the original force system is equivalent to a single force $\bar{F} = \Sigma \bar{F}$ whose point of application is Q.

Example 45

Note that *the point of application of a single equivalent force is not unique.* One can "slide" a single equivalent force along its line of application and obtain another equivalent force with a different point of application.

Suppose one has a force system which is equivalent to single force $\bar{F}$ placed at a point Q. Let $\bar{T} = \sum \bar{M}^O$ where O is any point. Then it is necessary that $\bar{T}$ is perpendicular to $\bar{F}$. This can be seen as follows. Since $\bar{F}$ placed at Q is equivalent to the original force system, the moment of $\bar{F}$ about O must equal the moment resultant of the original force system about O, that is,

$$\bar{r} \times \bar{F} = \bar{T} \tag{12.1}$$

where $\bar{r}$ is the vector from O to Q. The above expression and the properties tell that $\bar{T}$ is perpendicular to $\bar{F}$.

Actually, given any two mutually perpendicular vectors $\bar{F}$ and $\bar{T}$ with $\bar{F} \neq 0$, one can always find a vector $\bar{r}$ such that (12.1) holds. One such vector is given by

$$\bar{r} = \frac{\bar{F} \times \bar{T}}{F^2}\,. \tag{12.2}$$

This can be seen as follows. Recall that for any three vectors $\bar{U}$, $\bar{V}$ and $\bar{W}$ we have

$$\bar{U} \times (\bar{V} \times \bar{W}) = (\bar{U} \cdot \bar{W})\bar{V} - (\bar{U} \cdot \bar{V})\bar{W}\,.$$

Hence

$$\bar{r} \times \bar{F} = \left(\frac{\bar{F} \times \bar{T}}{F^2}\right) \times \bar{F} = -\frac{1}{F^2}\left(\bar{F} \times (\bar{F} \times \bar{T})\right) = -\frac{1}{F^2}\left((\bar{F} \cdot \bar{T})\bar{F} - (\bar{F} \cdot \bar{F})\bar{T}\right)\,.$$

Since, by assumption, $\bar{F}$ is perpendicular to $\bar{T}$ we must have $\bar{F} \cdot \bar{T} = 0$. Also, $\bar{F} \cdot \bar{F} = F^2$. Hence, we obtain that $\bar{r} \times \bar{F} = \bar{T}$.

The following force systems are examples of force systems which are equivalent to a single force.

Concurrent force system

Here the line of each force passes through a common point.

Planar force systems with non-zero resultant

Example 46

Parallel force systems with non-zero resultant

Example 47

Parallel force systems. Choose a reference frame so that all the forces are parallel to $\hat{e}_3$. Then every force $\bar{F}^j$ can be expressed as

$$\bar{F}^j = F^j \, \hat{e}_3$$

and

$$\Sigma \bar{F} = F \hat{e}_3 \qquad \text{where} \qquad F := \sum_{i=1}^{N} F^j$$

Let O be the origin of the reference frame and let r^j be the vector from O to the point of application of F^j; then r^j can be expressed as

$$\bar{r}^j = x^j \, \hat{e}_1 + y^j \, \hat{e}_2 + z^j \, \hat{e}_3$$

Hence,

$$\bar{r}^j \times \bar{F}^j = (y^j F^j) \, \hat{e}_1 - (x^j F^j) \, \hat{e}_2$$

and

$$\Sigma \bar{M}^O = T_1 \, \hat{e}_1 + T_2 \, \hat{e}_2 \qquad \text{where} \qquad T_1 := \sum_{i=1}^{N} y^j F^j \quad \text{and} \quad T_2 := -\sum_{i=1}^{N} x^j F^j \, .$$

Letting

$$\bar{r}^* = x^* \, \hat{e}_1 + y^* \, \hat{e}_2 + z^* \, \hat{e}_3$$

be the vector from O to the point of application of $\bar{F}$ we get

$$\bar{r}^* \times \bar{F} = (y^* \sum_{i=1}^{N} F^j) \, \hat{e}_1 - (x^* \sum_{i=1}^{N} F^j) \, \hat{e}_2$$

Since $\bar{r}^* \times \bar{F} = \Sigma \bar{M}^O$, we obtain

$$\begin{aligned} y^* \sum_{i=1}^{N} F^j &= \sum_{i=1}^{N} y^j F^j \\ x^* \sum_{i=1}^{N} F^j &= \sum_{i=1}^{N} x^j F^j \end{aligned}$$

Hence

$$\boxed{\begin{aligned} x^* &= \frac{\sum_{i=1}^{N} x^j F^j}{F} \\ y^* &= \frac{\sum_{i=1}^{N} y^j F^j}{F} \end{aligned}}$$

12.8 Distributed force systems

So far, we have considered forces to act at a single point. Here we look at forces which do not act a single point, but act over a region of space. We divide these forces into body forces and surface forces.

12.8.1 Body forces

A body force acts over a three-dimensional region of space. The main example of a body force is the gravitational attraction of one body on another.

Gravitational forces

The gravitational force exerted by one body (the offending body) on another body (the suffering body) is a force system which is distributed throughout the entire suffering body. Every particle of the suffering body is subject to the gravitational attraction of the offending body. When the offending body is YFHB and the suffering body is near the surface of YFHB and its dimensions are insignificant compared to YFHB then, all the gravitational force system is in the same direction, namely, in the direction of the local vertical $\hat{g}$. So this gravitational force system is a parallel force system. It is equivalent to a single force $\bar{W}$ placed at the mass center of the suffering body and given by

$$\bar{W} = W\hat{g}$$

where W is called the weight of the suffering body and is given by

$$W = mg$$

where m is the mass of the suffering body and g is the gravitational acceleration constant of YFHB. For bodies of uniform mass density, the mass center is at the geometric center.

Figure 12.11: Weight

When the suffering body is not near the surface of YFHB (think of a spacecraft in orbit around the earth) then, the gravitational forces on the suffering body may have a non-zero moment resultant about the mass center of the suffering body.

Figure 12.12: Some mass centers

12.8.2 Surface forces

A surface force acts over a two-dimensional region of space. One example of a surface force is hydrostatic force, that is, the forces exerted on a body when it is immersed in water. Another example is aerodynamic forces, that is the forces on a body moving relative to the air.

Hydrostatic forces

Archimedes principle, center of buoyancy

Aerodynamic forces

lift, drag, pitching moment, center of pressure

Connection forces

The force exerted by one body on another at a connection between the two bodies is a distributed force system. We usually represent this force system by an equivalent force system consisting of a single force $\bar{R}$ and a couple of torque $\bar{T}$.

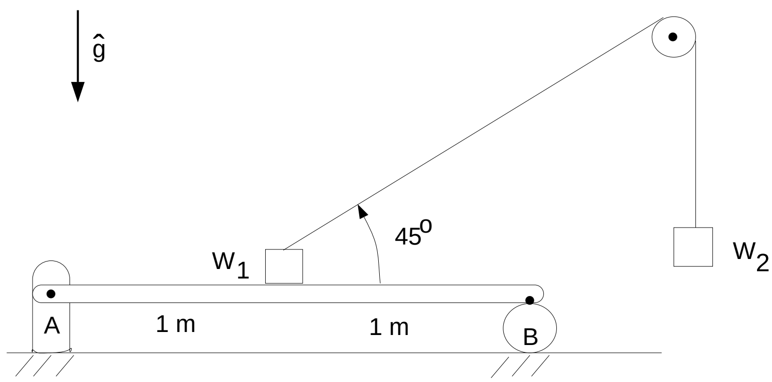
If the connection is smooth and permits translational motion in a specific direction then, the force $\bar{R}$ has no component in that direction. Conversely, if the connection prevents translational motion in a specific direction then, $\bar{F}$ can have a component in that direction.

If the connection is smooth and permits rotational motion about a specific axis then, the torque $\bar{T}$ *has no component about that axis. Conversely, if the connection prevents rotational motion about a specific axis then,* $\bar{T}$ *can have a component about that axis.*

12.9 More examples in static equilibrium

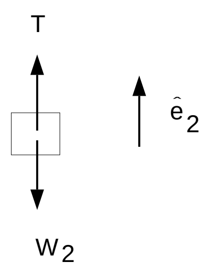
Example 48 The following system is in static equilibrium with

$$W_1 = 50\text{N} \qquad \text{and} \qquad W_2 = 60\text{N}\,.$$



Find the reactions on the bar at A and B.

Solution. Consider first the equilibrium of the block of weight W_2.



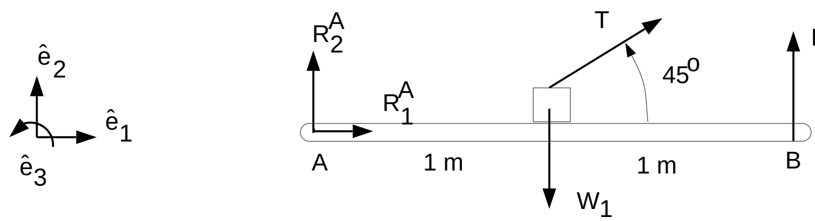
Considering $\Sigma\bar{F} = 0$ we obtain

$$\hat{e}_2 : -W_2 + T = 0 \tag{12.3}$$

Hence,

$$T = W_2 = 60\,\text{N}$$

Consider now the equilibrium of bar plus block of weight W_1.



Considering $\Sigma \bar{F} = 0$ we obtain

$$\hat{e}_1 \quad : \quad R_1^A + T \cos 45^o = 0 \tag{12.4}$$

$$\hat{e}_2 \quad : \quad R_2^A + T \sin 45^o - W_1 + R^B = 0 \tag{12.5}$$

Considering $\Sigma \bar{M}^A = 0$ we obtain

$$\hat{e}_3 : -W_1 + \sin 45^o T + 2R^B = 0 \tag{12.6}$$

We can now use these last three equations to solve for

$$\begin{aligned}
R^B &= (W_1 - T \sin 45^o)/2 &&= 3.787\text{N} \\
R_1^A &= -T \cos 45^o &&= -42.43\text{N} \\
R_2^A &= -T \sin 45^o + W_1 - R^B &&= 3.787\text{N}
\end{aligned}$$

$$\boxed{\begin{aligned}
\bar{R}^A &= -42.43\,\hat{e}_1 + 3.787\,\hat{e}_2 \quad \text{N} \\
\bar{R}^B &= 3.787\,\hat{e}_2 \quad \text{N}
\end{aligned}}$$

12.9.1 Two force bodies in static equilibrium

Consider a body which is subject to only two forces.

Figure 12.13: A two force body

Suppose that this body is in static equilibrium. Then, consideration of $\Sigma \bar{F} = \bar{0}$ tells us that the sum of these two forces must be zero. Thus, one force is the negative of the other; hence, *the two forces have the same magnitude, but, opposite direction.*

Consideration of $\Sigma \bar{M}^Q = \bar{0}$ where Q is the point of application of one of the forces tells us that this point must also be on the line of application of the other force; since the two forces are parallel, *they have the same line of application.*

Figure 12.14: Two force bodies in static equilibrium

Example 49

Example 50

12.10 Statically indeterminate problems

A problem is statically indeterminate if one cannot solve for all the unknown forces and torques using only the conditions of static equilibrium: $\Sigma \bar{F} = \bar{0}$ and $\Sigma M^Q = \bar{0}$.

Example 51

12.11 Internal forces

12.12 Exercises

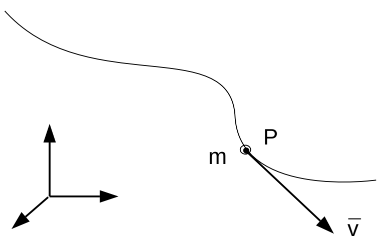
Exercise 29 Determine the reaction on the massless structure at pin joint A.

Exercise 30 Obtain a simple force system consisting of a single force or a single couple which is equivalent to the force system shown.

Chapter 13

Momentum

13.1 Linear momentum



Suppose we are observing the motion of a particle of mass m moving with velocity $\bar{v}$ relative to some inertial reference frame.

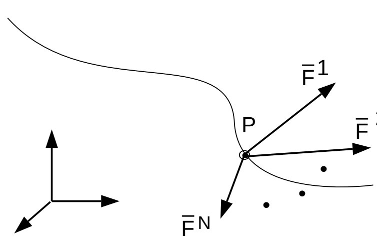
• The *linear momentum* of the particle in the inertial frame is defined by

$$\boxed{\overline{L} = m\bar{v}}$$

$\dim[\overline{L}] = \mathrm{MLT}^{-1} = \mathrm{FT}$

units: kg ms^{-1} or lb sec

Suppose $\Sigma\bar{F}$ is the sum of all the forces acting on the particle, i.e., $\Sigma\overline{F} = \sum_{j=1}^{N} \overline{F}^j$, where $\overline{F}^1, \overline{F}^2, \ldots, \overline{F}^N$ are all the forces acting on the particle.



Then, using $\Sigma\bar{F} = m\bar{a}$ we can obtain the following result for any inertial reference frame:

• *The sum of the forces acting on a particle equals the time rate of change of the linear momentum of the particle, i.e.,*

$$\boxed{\sum \overline{F} = \dot{\overline{L}}}$$

Proof. Since m is constant,

$$\begin{aligned} \dot{\bar{L}} &= \frac{d}{dt}(m\bar{v}) \\ &= m\frac{d}{dt}(\bar{v}) \\ &= m\bar{a} \end{aligned}$$

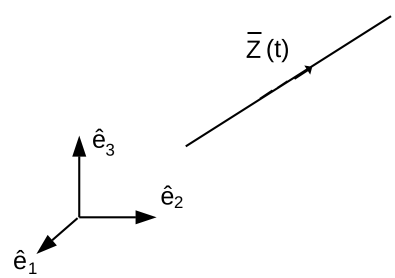
where $\bar{a}$ is the inertial acceleration of the particle. It now follows from $\Sigma\bar{F} = m\bar{a}$ that

$$\Sigma\overline{F} = \dot{\overline{L}}$$

■

13.2 Impulse of a force

13.2.1 Integral of a vector valued function



Suppose $\bar{Z}$ is a function of a scalar variable t and

$$\bar{Z} = Z_1\hat{e}_1 + Z_2\hat{e}_2 + Z_3\hat{e}_3$$

Consider any interval $[t_0,\, t_1]$. Relative to reference frame e, the integral of $\bar{Z}$ over $[t_0,\, t_1]$ is defined by

$$\int_{t_0}^{t_1} \bar{Z}\,dt := \left(\int_{t_0}^{t_1} Z_1\,dt\right)\hat{e}_1 + \left(\int_{t_0}^{t_1} Z_2\,dt\right)\hat{e}_2 + \left(\int_{t_0}^{t_1} Z_3\,dt\right)\hat{e}_3$$

Example 52 Suppose

$$\bar{Z}(t) = \hat{e}_1 + t\hat{e}_2 + \sin t\hat{e}_3$$

Then

$$\begin{aligned}\int_0^1 \bar{Z}\,dt &= \left(\int_0^1 1\,dt\right)\hat{e}_1 + \left(\int_0^1 t\,dt\right)\hat{e}_2 + \left(\int_0^1 \sin t\,dt\right)\hat{e}_3 \\ &= \hat{e}_1 + \frac{1}{2}\hat{e}_2 + (1-\cos(1))\hat{e}_3\end{aligned}$$

- Note that

$$\int_{t_0}^{t_1} \dot{\bar{Z}}\,dt = \bar{Z}(t_1) - \bar{Z}(t_0)$$

13.2.2 Impulse

- The *impulse* of a force $\bar{F}$ over a time interval $[t_0,\, t_1]$ is defined by

$$\boxed{\bar{I}_{t_0}^{t_1} := \int_{t_0}^{t_1} \bar{F}\,dt}$$

Impulsive forces. Large magnitude over a short time interval. They usually occur during impacts and collisions.

Suppose we are observing the motion of a particle relative to some inertial reference frame over some time interval $[t_0,\, t_1]$. Let $\Delta\overline{L}$ be the change in the linear momentum of the particle over the time interval, i.e,

$$\Delta\overline{L} = \overline{L}(t_1) - \overline{L}(t_0)$$

Let $\Sigma\bar{I}$ be the *total impulse* acting on the particle over the time interval, i.e, $\Sigma\bar{I}$ is the sum of the impulses of all the forces acting on the particle:

$$\Sigma\overline{I} = \sum_{j=1}^{N}\left(\int_{t_0}^{t_1} \overline{F}^j\,dt\right)$$

Then we have the following result for any inertial reference frame:

- *Over any time interval, the total impulse acting on a particle equals the change in the linear momentum of a particle, i.e.,*

$$\boxed{\Sigma\bar{I} = \Delta\bar{L}}$$

Proof. Recall that

$$\Sigma\overline{F} = \dot{\bar{L}}$$

Integrate over the time interval $[t_0,\, t_1]$ to yield:

$$\int_{t_0}^{t_1} \Sigma\overline{F}\,dt = \int_{t_0}^{t_1} \dot{\overline{L}}\,dt$$

We have

$$\begin{aligned}
\int_{t_0}^{t_1} \Sigma\overline{F}\,dt &= \int_{t_0}^{t_1} \left(\sum_{j=1}^{N} \overline{F}^j\right) dt \\
&= \sum_{j=1}^{N} \left(\int_{t_0}^{t_1} \overline{F}^j\,dt\right) \\
&= \Sigma\bar{I}
\end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{t_1} \dot{\overline{L}}\,dt &= \overline{L}(t_1) - \overline{L}(t_0) \\
&= \Delta\overline{L}
\end{aligned}$$

This yields the desired result. ■

We have the following immediate consequence of the above result.

• If the total impulse acting on a particle over any time interval $[t_0,\, t_1]$ is zero, then the linear momentum of the particle is *conserved* over that interval, i.e.,

$$\bar{L}(t_1) = \bar{L}(t_0)$$

From this it follows that the velocity is also conserved, i.e.,

$$\bar{v}(t_1) = \bar{v}(t_0)$$

This is also true for any compoment, i.e, if the total impulse has zero component in some direction, then the corresponding components of the linear momentum and the velocity are conserved; see next example.

Example 53

Ball impacts smooth wall. *Find* the exit speed v.

SOLUTION. Suppose the impact between wall and ball occurs over the interval $[t_0, t_1]$.

Looking at a FBD of the ball during impact, the only force with a vertical component is weight. Its impulse is

$$\int_{t_0}^{t_1} mg\hat{g}\, dt = mg(t_1 - t_0)\hat{g}$$

Ideally, we can choose $t_1 - t_0$ arbitrarily small; so the impulse due ro the weight force is negligible. Hence, the vertical component of the total impulse is zero. This implies that the vertical component of the ball's velocity is preserved during impact, i.e.,

$$v\cos(30°) = 10\cos(60°)$$

Hence,

$$\boxed{v = \frac{10}{\sqrt{3}}\,\mathrm{ms}^{-1}}$$

13.3 Angular momentum

Suppose we are observing the motion of a particle of mass m relative to some inertial reference frame i and Q is any point. Let $\bar{r}$ be the position of particle relative to Q and let $\dot{\bar{r}}$ be its time derivative in i.

- The *angular momentum* of the particle about Q is defined by

$$\boxed{\bar{H}^Q = \bar{r} \times m\dot{\bar{r}}}$$

$\dim[\overline{H}^Q] = \mathrm{ML^2T^{-1}} = \mathrm{FLT}$

units: kg m^2 s^{-1} or lb ft sec

In the above definition, Q does not have to be a fixed point. Suppose $Q = O$ where O is a point fixed in the inertial reference frame; then

$$\dot{\bar{r}} = \overline{v}$$

where $\bar{v}$ is the velocity of the particle in the frame; hence

$$\begin{aligned} \bar{H}^O &= \bar{r} \times m\bar{v} \\ &= \bar{r} \times \bar{L} \end{aligned}$$

where $\bar{L}$ is the momentum of the particle in the inertial frame.

Example 54 *Angular momentum and polar coordinates*

We have

$$\bar{r} = r\hat{e}^r$$

$$\dot{\bar{r}} = r\omega\,\hat{e}_\theta$$

Hence

$$\bar{H}^O = mr^2\omega\,\hat{e}_3$$

Suppose $\Sigma\bar{M}^O$ is the sum of the moments about point O of all the forces acting on the particle, i.e.,

$$\begin{aligned}\Sigma\bar{M}^O &:= \sum_{j=1}^{N}\bar{r}\times\overline{F}^j \\ &= \bar{r}\times\sum_{j=1}^{N}\overline{F}^j \\ &= \bar{r}\times\Sigma\bar{F}\end{aligned}$$

where $\bar{r}$ is the position of the particle relative to O. We have now the following result for any inertial reference frame:

- *The time rate of change of the angular momentum of a particle about a fixed point O is equal to the sum of the moments about O of all the forces acting on the particle, i.e.,*

$$\boxed{\sum\overline{M}^O = \dot{\overline{H}}^O}$$

Proof. Since m is constant,

$$\begin{aligned}\dot{\overline{H}}^O &= \frac{d\bar{H}^O}{dt} \\ &= \frac{d}{dt}(\bar{r}\times m\dot{\bar{r}}) \\ &= \dot{\bar{r}}\times m\dot{\bar{r}} + \bar{r}\times m\ddot{\bar{r}} \\ &= \bar{r}\times m\ddot{\bar{r}}\end{aligned}$$

Since O is fixed in frame i, the vector $\ddot{\bar{r}}$ equals $\bar{a}$, the inertial acceleration of the particle in i; since i is inertial we have $\Sigma\bar{F} = m\bar{a}$, hence

$$\begin{aligned}\dot{\overline{H}}^O &= \bar{r}\times\Sigma\bar{F} \\ &= \Sigma\bar{M}^O\end{aligned}$$

and we obtain the desired result. ■

Example 55 *Simple pendulum*

13.4 Central force motion

Recall that a particle undergoes central force motion if is subject to a single force $\overline{F}$ whose line of action always passes through some inertially fixed point O.

Chapter 14

Work and Energy

14.1 Kinetic energy

Consider the motion of a *particle* of mass m relative to some inertial reference frame. The kinetic energy of the particle is denoted by T and is given by

$$T = \frac{1}{2}mv^2$$

where v is the speed of the particle and m is the mass of the particle. Note that kinetic energy is a scalar quantity. If $\bar{v}$ is the velocity of the particle, we may also express T as

$$T = \frac{1}{2}m\bar{v} \cdot \bar{v}\,.$$

Units
SI: Joule (J); J = N m = kg $\mathrm{m}^2/\mathrm{s}^2$
Other: lb ft, btu

Example 56 (Kinetic energy of pendulum)

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

Consider now a *rigid body* of mass m which is *translating* with speed v relative to some inertial reference frame. The kinetic energy of the body is simply the sum of the kinetic energies of the particles composing the body. Hence the kinetic energy of a translating rigid body is also given by $T = \frac{1}{2}mv^2$ where m is the mass of the body.

14.2 Power

Consider a force $\bar{F}$ acting on a particle which is moving with velocity $\bar{v}$ relative to some inertial reference frame. The power of the force is denoted by $\mathcal{P}$ and is given by

$$\mathcal{P} = \bar{F} \cdot \bar{v}$$

Example 57 Normal forces and friction

Consider now a *rigid body* which is *translating* with velocity $\bar{v}$ relative to some inertial reference frame. Suppose it is subject to a distributed force system whose resultant is $\bar{F}$. Then the power of the distributed force system is defined to be the resultant of the powers of the forces which make up the force system. This power is also given by $\mathcal{P} = \bar{F} \cdot \bar{v}$.

Example 58 Lift and drag

14.3 A basic result

Consider a particle in motion relative to an inertial reference frame. We define the resultant power of all the forces acting on the particle to be the sum of the powers of all the forces acting on the particle. We have now the following result.

The resultant power of all the forces acting on a particle equals the time rate of change of the kinetic energy of the particle.

Mathematically, we can represent the above result as

$$\boxed{\sum \mathcal{P} = \dot{T}}$$

where $\sum \mathcal{P}$ is the resultant power of all the forces acting on the particle and T is the kinetic energy of the particle.

Example 59 (The simple pendulum) Show that

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

14.4 Conservative forces and potential energy

Consider a force acting on a particle moving in some inertial reference frame. We say that this force is conservative, if there is a scalar function ϕ with the following properties:

(a) The function ϕ is only a function of the position of the particle relative to some inertially fixed point.

(b) The power of the force is the negative of the time rate of change of ϕ, that is

$$\mathcal{P} = -\dot{\phi}$$

where $\mathcal{P}$ is the power of the force.

The function ϕ is called the potential energy of the force. So, we have the following ststement.

The power of a conservative force is the negative of the time rate of change of its potential energy.

14.4.1 Weight

$$\phi = mgh$$

14.4.2 Linear springs

$$\phi = \frac{1}{2}k(l - l_0)^2$$

14.4.3 Inverse square gravitational force

$$\phi = -\frac{GMm}{r}$$

14.5 Total mechanical energy

Consider a particle in motion relative to some inertial reference frame. We can divide the forces acting on the particle into conservative forces and non-conservative forces. Associated with each conservative force is a potential energy. Let ϕ be the sum of the potential energies of all the conservative forces. We call this the total potential energy. The total mechanical energy is the sum of the kinetic energy and the total potential energy, that is

$$\boxed{E = T + \phi}$$

where E denotes the total mechanical energy.

Let $\sum^c \mathcal{P}$ be the resultant power of all the conservative forces. Since the power of a conservative force equals the negative of the time rate of change of its potential energy, it

follows that the resultant power of the conservative forces equals the negative of the time rate of change of the total potential energy, that is,

$$\sum^{c} \mathcal{P} = -\dot{\phi}\,.$$

Let $\sum^{nc} \mathcal{P}$ demote the resultant power of all the non-conservative forces. Then the resultant power $\sum \mathcal{P}$ of all the forces satisfies

$$\sum \mathcal{P} = \sum^{c} \mathcal{P} + \sum^{nc} \mathcal{P} = -\dot{\phi} + \sum^{nc} \mathcal{P}\,.$$

Recalling that $\sum \mathcal{P} = \dot{T}$, we have

$$-\dot{\phi} + \sum^{nc} \mathcal{P} = \dot{T}\,.$$

Hence

$$\sum^{nc} \mathcal{P} = \dot{T} + \dot{\phi}$$

or

$$\boxed{\sum^{nc} \mathcal{P} = \dot{E}}\,.$$

The resultant power of all the nonconservative forces equals the time rate of change of the total mechanical energy.

So, if the resultant power of the nonconservative forces is zero, then the time rate of change of the mechanical energy is zero. In this case the energy E is constant and we say that *the total mechanical energy is conserved.*

Example 60 (The simple pendulum) Show that

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

14.6 Work

Basically, work is the time integral of power. More specifically, consider any time interval $[t_1, t_2]$. Then the work done by a force over that interval is denoted by WD is given by

$$WD = \int_{t_1}^{t_2} \mathcal{P}\, dt$$

where $\mathcal{P}$ is the power of the force.

We have now the following results:

$$\sum WD = \Delta T$$

where $\Delta T = T(t_2) - T(t_1)$.

Also,

$$\sum^{nc} WD = \Delta E$$

where $\Delta E = E(t_2) - E(t_1)$.

Example 61

Chapter 15

Systems of Particles